# Crystallized Peter-Weyl type decomposition for level 0 part of modified quantum algebra $\widetilde{U}_q(\widehat{\mathfrak{sl}_2})_0$

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## 1 Introduction

Modified quantum algebra $\widetilde{U}_q(\mathfrak{g})$ is an algebra which is 'modified' the Cartan part of the underlying quantum algebra $U_q(\mathfrak{g})$ as in (3.1) and (3.2). Modified quantum algebra $\widetilde{U}_q(\mathfrak{g})$ holds the remarkable property that modified quantum algebra $\widetilde{U}_q(\mathfrak{g})$ affords the commutative two crystal structures. The first one is the usual crystal structure, which was found by Lusztig ([8]). Another crystal structure was discovered by Kashiwara ([4]), which is called right crystal structure. In [4], it is shown that $\widetilde{U}_q(\mathfrak{g})$ is stable by the action of the antiautomorphism $*$ (see 2.1) and moreover, crystal base $B(\widetilde{U}_q(\mathfrak{g}))$ is also stable by the action of $*$. By using $*$, right crystal structure is constructed. The commutativity of those crystal structures motivated us to consider Peter-Weyl type decomposition on the crystal base of $\widetilde{U}_q(\mathfrak{g})$.

In [4], Kashiwara gave the Peter-Weyl type decomposition for the crystal base of modified quantum algebra of finite type and affine type of non-zero level part. But the Peter-Weyl type decomposition for affine type with level 0 part is still unclear. In this paper, we shall give some criteria for the existence of the Peter-Weyl type decomposition:

$$B(\widetilde{U}_q(\mathfrak{g})) \cong \bigoplus_{\lambda\in P/W} B^{\max}(\lambda)\otimes B(-\lambda)^*,$$

where $P$ is a weight lattice, $W$ is the Weyl group associated with $\mathfrak{g}$ and $B^{\max}(\lambda)$ and $B(-\lambda)^*$ will be given in section 4. Those criteria are related to the property of connected component in $B(\widetilde{U}_q(\mathfrak{g}))$. Furthermore, we can consider its application to the level 0 part of modified quantum affine algebra $\widetilde{U}_q(\widehat{\mathfrak{sl}_2})_0$ since we

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have already classified all connected components in $\widetilde{U}_q(\widehat{\mathfrak{sl}_2})_0$ in [10] and got the properties required by the criteria.

Let us see the organization of this paper. In section 2, we prepare the notion of crystals and related subjects. In section 3, we review the definition of modified quantum algebras, the properties of their crystal base and general feature of operation $*$ and Weyl group on $B(\widetilde{U}_q(\mathfrak{g}))$. In this section, we shall give the definition of extremal vector. In section 4, we shall give the right structure on $B(\widetilde{U}_q(\mathfrak{g}))$. Then we shall investigate the criteria for the existence of the Peter-Weyl type decomposition. In section 5, we consider the application of the criteria to the level 0 part of $B(\widetilde{U}_q(\widehat{\mathfrak{sl}_2}))$. In order to give the explicit form of $B^{\max}(\lambda)$ and $B(-\lambda)^*$, we describe the action of $*$ on extremal vectors.

The author would like to acknowledge professor Masaki Kashiwara for his helpful advises. This work was partly done during the stay of the author at Northeastern University. He is grateful to professor Andrei Zelevinsky for his kind hospitality.

# 2 Crystals

## 2.1 Definition of $U_q(\mathfrak{g})$

Let $\mathfrak{g}$ be a symmetrizable Kac-Moody algebra over $\mathbf{Q}$ with a Cartan subalgebra $\mathfrak{t}$, $\{\alpha_i \in \mathfrak{t}^*\}_{i\in I}$ the set of simple roots and $\{h_i \in \mathfrak{t}\}_{i\in I}$ the set of coroots, where $I$ is a finite index set. We define an inner product on $\mathfrak{t}^*$ such that $(\alpha_i,\alpha_i) \in \mathbf{Z}_{\geq 0}$ and $\langle h_i, \lambda\rangle = 2(\alpha_i,\lambda)/(\alpha_i,\alpha_i)$ for $\lambda \in \mathfrak{t}^*$. Set $Q = \oplus_i \mathbf{Z}\alpha_i$, $Q_+ = \oplus_i \mathbf{Z}_{\geq 0}\alpha_i$ and $Q_- = -Q_+$. We call $Q$ a root lattice. Let $P$ a lattice of $\mathfrak{t}^*$ *i.e.* a free $\mathbf{Z}$-submodule of $\mathfrak{t}^*$ such that $\mathfrak{t}^* \cong \mathbf{Q} \otimes_{\mathbf{Z}} P$, and $P^* = \{h \in \mathfrak{t} | \langle h, P\rangle \subset \mathbf{Z}\}$. We set $P_+ = \{\lambda \in P | \langle \lambda, h_i\rangle \geq 0 \text{ for any } i \in I\}$. An element of $P$(resp.$P_+$) is called a integral weight (resp. dominant integral weight).

The quantized enveloping algebra $U_q(\mathfrak{g})$ is an associative $\mathbf{Q}(q)$-algebra generated by $e_i$, $f_i (i \in I)$ and $q^h (h \in P^*)$ satisfying the following relations:

$$q^0 = 1, \quad \text{and} \quad q^h q^{h'} = q^{h+h'}, \tag{2.1}$$

$$q^h e_i q^{-h} = q^{\langle h,\alpha_i\rangle} e_i, \qquad q^h f_i q^{-h} = q^{-\langle h,\alpha_i\rangle} f_i, \tag{2.2}$$

$$[e_i, f_j] = \delta_{i,j}(t_i - t_i^{-1})/(q_i - q_i^{-1}), \tag{2.3}$$

$$\sum_{k=1}^{1-\langle h_i,\alpha_j\rangle} (-1)^k x_i^{(k)} x_j x_i^{(1-\langle h_i,\alpha_j\rangle - k)} = 0, \qquad (i \neq j) \tag{2.4}$$

where $x_i = e_i, f_i$ and we set $q_i = q^{(\alpha_i,\alpha_i)/2}$, $t_i = q_i^{h_i}$, $[n]_i = (q_i^n - q_i^{-n})/(q_i - q_i^{-1})$, $[n]_i! = \prod_{k=1}^n [k]_i$, $e_i^{(n)} = e_i^n/[n]_i!$ and $f_i^{(n)} = f_i^n/[n]_i!$.

It is well-known that $U_q(\mathfrak{g})$ has a Hopf algebra structure with a comultiplication $\Delta$ given by

$$\Delta(q^h) = q^h \otimes q^h, \quad \Delta(e_i) = e_i \otimes t_i^{-1} + 1 \otimes e_i, \quad \Delta(f_i) = f_i \otimes 1 + t_i \otimes f_i,$$

for any $i \in I$ and $h \in P^*$. We do not describe an antipode and a counit. By this comultiplication, a tensor product of $U_q(\mathfrak{g})$-modules has a $U_q(\mathfrak{g})$-module structure.
Let $*$ be the antiautomorphism of $U_q(\mathfrak{g})$ given by:

$$q^* = q, \quad (q^h)^* = q^{-h}, \quad e_i^* = e_i, \quad f_i^* = f_i. \tag{2.5}$$

Let $\vee$ be the automorphism of $U_q(\mathfrak{g})$ given by:

$$q^\vee = q, \quad (q^h)^\vee = q^{-h}, \quad e_i^\vee = f_i, \quad f_i^\vee = e_i. \tag{2.6}$$

These satisfy

$$* * = \vee\vee = \mathrm{id}, \qquad *\vee = \vee * . \tag{2.7}$$

## 2.2 Definition of Crystals

Let us recall the definition of crystals [3, 4]. The notion of a crystal is motivated by abstracting the some combinatorial properties of crystal bases.

**Definition 2.1** *A crystal $B$ is a set with the following data:*

$$\text{a map} \quad wt : B \longrightarrow P, \tag{2.8}$$

$$\varepsilon_i : B \longrightarrow \mathbf{Z} \sqcup \{-\infty\}, \quad \varphi_i : B \longrightarrow \mathbf{Z} \sqcup \{-\infty\}, \quad \text{for} \quad i \in I, \tag{2.9}$$

$$\tilde{e}_i : B \longrightarrow B \sqcup \{0\}, \quad \tilde{f}_i : B \longrightarrow B \sqcup \{0\} \quad \text{for} \quad i \in I. \tag{2.10}$$

*Here 0 is an ideal element which is not included in $B$. They are subject to the following axioms: For $b$,$b_1$,$b_2 \in B$,*

$$\varphi_i(b) = \varepsilon_i(b) + \langle h_i, wt(b)\rangle, \tag{2.11}$$

$$wt(\tilde{e}_i b) = wt(b) + \alpha_i \ \text{if } \tilde{e}_i b \in B, \tag{2.12}$$

$$wt(\tilde{f}_i b) = wt(b) - \alpha_i \ \text{if } \tilde{f}_i b \in B, \tag{2.13}$$

$$\tilde{e}_i b_2 = b_1 \ \text{if and only if } \tilde{f}_i b_1 = b_2, \tag{2.14}$$

$$\text{if } \varepsilon_i(b) = -\infty, \ \text{then } \tilde{e}_i b = \tilde{f}_i b = 0. \tag{2.15}$$

From the axiom (2.14), we can consider the graph strucure on a crystal $B$.

**Definition 2.2** *The crystal graph of crystal $B$ is an oriented and colored graph given by the rule : $b_1 \xrightarrow{i} b_2$ if and only if $b_2 = \tilde{f}_i b_1$ $(b_1, b_2 \in B)$.*

**Definition 2.3** *(i) If $B$ has the weight decomposition $B = \bigsqcup_{\lambda \in P} B_\lambda$ where $B_\lambda = \{b \in B | wt(b) = \lambda\}$ for $\lambda \in P$, we call $B$ a $P$-weighted crystal.*

(*ii*) *Let $B_1$ and $B_2$ be crystals. A morphism of crystals $\psi : B_1 \longrightarrow B_2$ is a map $\psi : B_1 \sqcup \{0\} \longrightarrow B_2 \sqcup \{0\}$ satisfying the following axioms:*

$$\psi(0) = 0, \tag{2.16}$$

$$wt(b) = wt(\psi(b)), \quad \varepsilon_i(b) = \varepsilon_i(\psi(b)), \quad \varphi_i(b) = \varphi_i(\psi(b)) \quad \text{if } b \in B_1 \text{ and } \psi(b) \in B_2, \tag{2.17}$$

$$\psi(\tilde{e}_i b) = \tilde{e}_i \psi(b) \text{ if } b \in B_1 \text{ satisfies } \psi(b) \neq 0 \text{ and } \psi(\tilde{e}_i b) \neq 0, \tag{2.18}$$

$$\psi(\tilde{f}_i b) = \tilde{e}_i \psi(b) \text{ if } b \in B_1 \text{ satisfies } \psi(b) \neq 0 \text{ and } \psi(\tilde{f}_i b) \neq 0. \tag{2.19}$$

(*iii*) *A morphism of crystals $\psi : B_1 \longrightarrow B_2$ is called strict if the associated map from $B_1 \sqcup \{0\} \longrightarrow B_2 \sqcup \{0\}$ commutes with all $\tilde{e}_i$ and $\tilde{f}_i$. If $\psi$ is injective, surjective and strict, $\psi$ is called an isomorphism.*

(*iv*) *A crystal $B$ is a normal, if for any subset $J$ of $I$ such that $((\alpha_i, \alpha_j))_{i,j \in J}$ is a positive symmetric matrix, $B$ is isomorphic to a crystal base of an integrable $U_q(\mathfrak{g}_J)$-module, where $U_q(\mathfrak{g}_J)$ is the quantum algebra generated by $e_j$, $f_j$ $(j \in J)$ and $q^h$ $(h \in P^*)$.*

For crystals $B_1$ and $B_2$, we shall define their tensor product $B_1 \otimes B_2$ as follows:

$$B_1 \otimes B_2 = \{b_1 \otimes b_2 | b_1 \in B_1, \, b_2 \in B_2\}, \tag{2.20}$$

$$wt(b_1 \otimes b_2) = wt(b_1) + wt(b_2), \tag{2.21}$$

$$\varepsilon_i(b_1 \otimes b_2) = \max(\varepsilon_i(b_1), \varepsilon_i(b_2) - \langle h_i, wt(b_1) \rangle), \tag{2.22}$$

$$\varphi_i(b_1 \otimes b_2) = \max(\varphi_i(b_2), \varphi_i(b_1) + \langle h_i, wt(b_2) \rangle), \tag{2.23}$$

$$\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2) \\ b_1 \otimes \tilde{e}_i b_2 & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases} \tag{2.24}$$

$$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2) \\ b_1 \otimes \tilde{f}_i b_2 & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2), \end{cases} \tag{2.25}$$

Here we understand that $0 \otimes b = b \otimes 0 = 0$. Let $\mathcal{C}(I, P)$ be the category of crystals determined by the data $I$ and $P$. Then $\otimes$ is a functor from $\mathcal{C}(I, P) \times \mathcal{C}(I, P)$ to $\mathcal{C}(I, P)$ and satisfies the associative law: $(B_1 \otimes B_2) \otimes B_3 \cong B_1 \otimes (B_2 \otimes B_3)$ by $(b_1 \otimes b_2) \otimes b_3 \leftrightarrow b_1 \otimes (b_2 \otimes b_3)$. Therefore, the category of crystals is endowed with the structure of tensor category.

For a crystal $B$, let $B^\wedge$ be the crystal given by

$$B^\wedge := \{b^\wedge | b \in B\},$$

$$wt(b^\wedge) = -wt(b), \qquad \varepsilon_i(b^\wedge) = \varphi_i(b), \qquad \varphi_i(b^\wedge) = \varepsilon_i(b),$$

$$\tilde{e}_i(b^\wedge) = (\tilde{f}_i b)^\wedge, \qquad \tilde{f}_i(b^\wedge) = (\tilde{e}_i b)^\wedge.$$

Then we have

$$(B_1 \otimes B_2)^\wedge \cong B_2^\wedge \otimes B_1^\wedge \qquad \text{by} \qquad (b_1 \otimes b_2)^\wedge \leftrightarrow b_2^\wedge \otimes b_1^\wedge. \tag{2.26}$$

**Example 2.4** *We give some examples of crystals.*

(i) *For* $\lambda \in P$, *we set* $T_\lambda = \{t_\lambda\}$ *with*

$$wt(t_\lambda) = \lambda, \quad \varepsilon_i(t_\lambda) = \varphi_i(t_\lambda) = -\infty, \quad \tilde{e}_i(t_\lambda) = \tilde{f}_i(t_\lambda) = 0.$$

*We can see that* $T_\lambda \otimes T_\mu \cong T_{\lambda+\mu}$ *and* $B \otimes T_0 \cong T_0 \otimes B \cong B$ *for any crystal* $B$.

(ii) *For* $i \in I$, *we set* $B_i := \{(n)_i \,|\, n \in \mathbf{Z}\}$ *with*

$$\begin{aligned}
&wt((n)_i) = n\alpha_i, \qquad \varepsilon_i((n)_i) = -n, \qquad \varphi_i((n)_i) = n,\\
&\varepsilon_j((n)_i) = -\infty, \qquad \varphi_j((n)_j) = -\infty \quad \text{for} \quad j \neq i,\\
&\tilde{e}_i(n)_i = (n+1)_i, \qquad \tilde{f}_i(n)_i = (n-1)_i,\\
&\tilde{e}_j(n)_i = \tilde{f}_j(n)_i = 0 \quad \text{for} \quad j \neq i.
\end{aligned}$$

(iii) *For* $\lambda \in P_+$, *let* $(L(\lambda), B(\lambda))$ *be the crystal base of a* $U_q(\mathfrak{g})$*-integrable highest weight module* $V(\lambda)$. $B(\lambda)$ *is the crystal associated with* $V(\lambda)$. *Set* $B(-\lambda) = B(\lambda)^\wedge$. *Then* $B(-\lambda)$ *is isomorphic to the crystal associated with an integrable lowest weight module* $V(-\lambda)$. *For* $b \in B(\lambda)$, $\varepsilon_i(b)$ *and* $\varphi_i(b)$ *are given by*

$$\begin{aligned}
\varepsilon_i(b) &= \max\{n | \tilde{e}_i^n b \neq 0\},\\
\varphi_i(b) &= \max\{n | \tilde{f}_i^n b \neq 0\}.
\end{aligned}$$

(iv) *Let* $(L(\infty), B(\infty))$ *be a crystal base of* $U_q^-(\mathfrak{g})$ $B(\infty)$ *is a crystal associated with* $U_q^-(\mathfrak{g})$. *Set* $B(-\infty) = B(\infty)^\wedge$. $B(-\infty)$ *is isomorphic to the crystal associated with* $U_q^+(\mathfrak{g})$. *In fact, we have* $B(-\infty) = B(\infty)^\wedge \cong B(\infty)^\vee$ *by* $b^\wedge \leftrightarrow b^\vee$. *For any* $b \in B(\infty)$ *and* $i \in I$ *there exists* $k$ *such that* $\tilde{e}_i^k b = 0$. *Then,* $\varepsilon_i(b)$ *and* $\varphi_i(b)$ *are given by*

$$\begin{aligned}
\varepsilon_i(b) &= \max\{n | \tilde{e}_i^n b \neq 0\},\\
\varphi_i(b) &= \langle h_i, wt(b)\rangle + \varepsilon_i(b).
\end{aligned}$$

# 3 Crystals of modified quantum algebra

This section is devoted to review [4],[9] (See also [8]).

## 3.1 Modified quantum algebra and Crystal base

For an integral weight $\lambda \in P$, let $U_q(\mathfrak{g})a_\lambda$ be the left $U_q(\mathfrak{g})$-module given by

$$U_q(\mathfrak{g})a_\lambda := U_q(\mathfrak{g}) / \sum_{h \in P^*} U_q(\mathfrak{g})(q^h - q^{\langle h, \lambda\rangle}), \tag{3.1}$$

where $a_\lambda$ is the image of the unit by the canonical projection. We set

$$\widetilde{U}_q(\mathfrak{g}) = \bigoplus_{\lambda\in P} U_q(\mathfrak{g})a_\lambda, \tag{3.2}$$

which is called *modified quantum algebra*.

We shall see a crystal base of $\widetilde{U}_q(\mathfrak{g})$. Taking $\lambda \in P$ and choosing $\zeta, \mu \in P_+$ such that $\lambda = \zeta - \mu$, we get the following $U_q(\mathfrak{g})$-linear surjective homomorphism:

$$\begin{array}{rcl} \pi_{\zeta,\mu} : U_q(\mathfrak{g})a_\lambda & \longrightarrow & V(\zeta)\otimes V(-\mu), \\ a_\lambda & \mapsto & u_\zeta \otimes u_{-\mu}. \end{array} \tag{3.3}$$

where $V(\zeta)$ and $V(-\mu)$ are as in Example 2.4 (iii) and $u_\zeta$ and $u_{-\mu}$ are their highest weight vector and lowest weight vector respectively.

**Theorem 3.1** (*cf* **[9]**) *For any $\lambda \in P$, there exists a unique pair $(L(U_q(\mathfrak{g})a_\lambda), B(U_q(\mathfrak{g})a_\lambda))$ which satisfies the following properties.*

(*i*) *We set $A := \{f(q) \in \mathbf{Q}(q) | f$ has no pole at $q = 0\}$. $L(U_q(\mathfrak{g})a_\lambda)$ is a free $A$-module such that $U_q(\mathfrak{g})a_\lambda \cong \mathbf{Q}(q) \otimes_A L(U_q(\mathfrak{g})a_\lambda)$ and $B(U_q(\mathfrak{g})a_\lambda)$ is a $\mathbf{Q}$-basis of the $\mathbf{Q}$-vector space $L(U_q(\mathfrak{g})a_\lambda)/qL(U_q(\mathfrak{g})a_\lambda)$.*

(*ii*) *For any $\zeta, \mu \in P_+$ with $\lambda = \zeta - \mu$, we have*

$$\pi_{\zeta,\mu}(L(U_q(\mathfrak{g})a_\lambda)) \subset L(\zeta) \otimes_A L(-\mu),$$

*and the induced map $\bar{\pi}_{\zeta,\mu}$:*

$$\bar{\pi}_{\zeta,\mu} \,:\, L(U_q(\mathfrak{g})a_\lambda)/qL(U_q(\mathfrak{g})a_\lambda) \longrightarrow (L(\zeta)/qL(\zeta)) \otimes (L(-\mu)/qL(-\mu)),$$

*satisfies $\bar{\pi}_{\zeta,\mu}(B(U_q(\mathfrak{g})a_\lambda)) \subset B(\zeta) \otimes B(-\mu) \sqcup \{0\}$.*

(*iii*) *There is a structure of crystal on $B(U_q(\mathfrak{g})a_\lambda)$ such that $\bar{\pi}_{\zeta,\mu}$ gives a strict morphism of crystals for any $\zeta, \mu \in P_+$ with $\lambda = \zeta - \mu$.*

Set

$$(L(\widetilde{U}_q(\mathfrak{g})), B(\widetilde{U}_q(\mathfrak{g}))) := \bigoplus_{\lambda\in P} (L(U_q(\mathfrak{g})a_\lambda), B(U_q(\mathfrak{g})a_\lambda)).$$

*Remark.* $B(U_q(\mathfrak{g})a_\lambda)$ is a normal crystal and then $B(\widetilde{U}_q(\mathfrak{g}))$ is a normal crystal.

Let $B(\infty)$, $B(-\infty)$ and $T_\lambda$ ($\lambda \in P$) be the crystals given in Example 2.4. The following theorem plays a significant role in this paper (See [4, Sec.3]).

**Theorem 3.2** $B(U_q(\mathfrak{g})a_\lambda) \cong B(\infty) \otimes T_\lambda \otimes B(-\infty)$ *as a crystal.*

**Corollary 3.3** $B(\widetilde{U}_q(\mathfrak{g})) \cong \oplus_{\lambda\in P} B(\infty) \otimes T_\lambda \otimes B(-\infty)$ *as a crystal.*

## 3.2 Description of the operation $*$

By the definition of the operation $*$ given in (2.5), it acts on an element of $\widetilde{U}_q(\mathfrak{g})$ as follows; Let $P$ be an element of $U_q^-(\mathfrak{g})U_q^+(\mathfrak{g})+U_q^+(\mathfrak{g})U_q^-(\mathfrak{g})$. Arbitrary element $u \in U_q(\mathfrak{g})a_\lambda$ can be written in the form $u = Pa_\lambda$. Then we have

$$(Pa_\lambda)^* = a_{-\lambda}P^*. \tag{3.4}$$

Furthermore, we have the following results:

**Theorem 3.4 ([4])** *(i) $L(\widetilde{U}_q(\mathfrak{g}))$ is invariant by $*$.*

*(ii) $B(\widetilde{U}_q(\mathfrak{g}))^* = B(\widetilde{U}_q(\mathfrak{g}))$.*

*(iii) For $\lambda \in P$, $b_1 \in B(\infty)$ and $b_2 \in B(-\infty)$, we get*

$$(b_1 \otimes t_\lambda \otimes b_2)^* = b_1^* \otimes t_{-\lambda - wt(b_1) - wt(b_2)} \otimes b_2^*. \tag{3.5}$$

## 3.3 Weyl group action and Extremal vectors

This subsection is devoted to review [4, Sec.7.8.9]. Let $B$ be a normal crystal (See Definition 2.3 (iv)). Let us define the Weyl group action on the underlying set $B$. For $i \in I$ and $b \in B$, we set

$$S_i b = \begin{cases} \tilde{f}_i^{\langle h_i, wt(b)\rangle} b & \text{if } \langle h_i, wt(b)\rangle \geq 0 \\ \tilde{e}_i^{-\langle h_i, wt(b)\rangle} b & \text{if } \langle h_i, wt(b)\rangle < 0. \end{cases} \tag{3.6}$$

We can easily obtain the following formula:

$$S_i^2 = \mathrm{id}, \qquad S_i\tilde{e}_i = \tilde{f}_i S_i, \qquad wt(S_i b) = s_i(wt(b)), \tag{3.7}$$

where $s_i(\lambda) = \lambda - \langle h_i, \lambda\rangle \alpha_i$ is the simple reflection.

Let $\mathfrak{g}$ be a rank 2 finite dimensional Lie algebra, and $W$ be the Weyl group associated with $\mathfrak{g}$. We set $w_0 = s_{i_1} \cdots s_{i_k}$ a reduced expression of the longest element of $W$. Here we get the following ([4, Sec.7]):

**Proposition 3.5** *Let $B$ be a normal crystal. For any $b \in B$, $S_{i_1} \cdots S_{i_k} b$ does not depend on the choice of reduced expression.*

**Corollary 3.6** *$\{S_i\}_{i=1,2}$ satisfies the braid relation.*

Thus for general $\mathfrak{g}$, we know that $\{S_i\}_{i \in I}$ defines the Weyl group action on a normal crystal $B$.

**Definition 3.7** *(i) Let $B$ be a normal crystal. An element $b \in B$ is called $i$-extremal, if $\tilde{e}_i b = 0$ or $\tilde{f}_i b = 0$.*

*(ii) An element $b \in B$ is called extremal if for any $l \geq 0$, $S_{i_1} \cdots S_{i_l} b$ is $i$-extremal for any $i$, $i_1 \cdots i_l \in I$.*

**Theorem 3.8** *Any connected component of $B(\widetilde{U}_q(\mathfrak{g}))$ contains an extremal vector.*

# 4 Criteria for Crystallized Peter-Weyl type decpomposition

## 4.1 Right crystal structure on $\widetilde{U}_q(\mathfrak{g})$.

Let $*$ be the antiautomorphism given in (2.5). By Theorem 3.4 (ii), we can define for $b \in B(\widetilde{U}_q(\mathfrak{g}))$;

$$\varepsilon_i^*(b) := \varepsilon_i(b^*), \qquad \varphi_i^*(b) := \varphi_i(b^*), \tag{4.1}$$

$$\tilde{e}_i^*(b) := (\tilde{e}_i(b^*))^*, \qquad \tilde{f}_i^*(b) := (\tilde{f}_i(b^*))^*, \tag{4.2}$$

By these, we can consider another crystal structure on $B(\widetilde{U}_q(\mathfrak{g}))$. This another crystal structure has the following remarkable property and that motivated us to consider the Peter-Weyl type decomposition on $B(\widetilde{U}_q(\mathfrak{g}))$.

**Theorem 4.1 ([4])** *$\tilde{e}_i^*$ and $\tilde{f}_i^*$ are commutative with all $\tilde{e}_i$'s and $\tilde{f}_i$'s on $B(\widetilde{U}_q(\mathfrak{g}))$, that is, on $B(\widetilde{U}_q(\mathfrak{g}))$ for any $i, j \in I$*

$$\tilde{e}_j\tilde{e}_i^* = \tilde{e}_i^*\tilde{e}_j, \quad \tilde{f}_j\tilde{e}_i^* = \tilde{e}_i^*\tilde{f}_j, \quad \tilde{e}_j\tilde{f}_i^* = \tilde{f}_i^*\tilde{e}_j, \quad \tilde{f}_j\tilde{f}_i^* = \tilde{f}_i^*\tilde{f}_j.$$

A crystal endowed with another crystal structure as above is called *bi-crystal*.

## 4.2 Crystal $B^{\max}(\lambda)$

For $\lambda \in P$, we set

$$B^{\max}(\lambda) := \{b \in B(U_q(\mathfrak{g})a_\lambda) \,|\, b^* \text{ is an extremal vector}\}. \tag{4.3}$$

By the following lemma, $B^{\max}(\lambda)$ is a subcrystal of $B(\widetilde{U}_q(\mathfrak{g}))$.

**Lemma 4.2**

$$\tilde{e}_iB^{\max}(\lambda) \subset B^{\max}(\lambda) \sqcup \{0\}, \qquad \tilde{f}_iB^{\max}(\lambda) \subset B^{\max}(\lambda) \sqcup \{0\}.$$

*Proof.* Let $b$ be an element of $B^{\max}(\lambda)$. For any $i, i_1, \cdots, i_k \in I$, by the definition of extremal vector, $b^*$ satisfies,

$$\tilde{e}_iS_{i_1}\cdots S_{i_k}b^* = 0 \quad \text{or} \quad \tilde{f}_iS_{i_1}\cdots S_{i_k}b^* = 0. \tag{4.4}$$

By operating $*$ on the both sides of (4.4) and by the fact $** = \mathrm{id}$, it follows that

$$\tilde{e}_i^*S_{i_1}^*\cdots S_{i_k}^*b = 0 \quad \text{or} \quad \tilde{f}_i^*S_{i_1}^*\cdots S_{i_k}^*b = 0, \tag{4.5}$$

where $S_i^*b := (S_i(b^*))^*$. This (4.5) is written by using only $\tilde{e}_i^*$'s and $\tilde{f}_i^*$'s. Thus, by Theorem 4.1, for any $j \in I$ we obtain

$$\tilde{e}_i^*S_{i_1}^*\cdots S_{i_k}^*\tilde{e}_jb = 0 \quad \text{or} \quad \tilde{f}_i^*S_{i_1}^*\cdots S_{i_k}^*\tilde{e}_jb = 0$$

and

$$\tilde{e}_i^* S_{i_1}^* \cdots S_{i_k}^* \tilde{f}_j b = 0 \quad \text{or} \quad \tilde{f}_i^* S_{i_1}^* \cdots S_{i_k}^* \tilde{f}_j b = 0.$$

Therefore, by operating $*$ on the both sides, we get

$$\tilde{e}_i S_{i_1} \cdots S_{i_k} (\tilde{e}_j b)^* = 0 \quad \text{or} \quad \tilde{f}_i S_{i_1} \cdots S_{i_k} (\tilde{e}_j b)^* = 0$$

and

$$\tilde{e}_i S_{i_1} \cdots S_{i_k} (\tilde{f}_j b)^* = 0 \quad \text{or} \quad \tilde{f}_i S_{i_1} \cdots S_{i_k} (\tilde{f}_j b)^* = 0.$$

We have completed the proof. □

The crystal $B^{\max}(\lambda)$ has the following properties:

**Proposition 4.3 ([4])** *(i) For any $i \in I$, $S_i^*$ gives an isomorphism;*

$$\begin{array}{rcl} S_i^* : B^{\max}(\lambda) & \xrightarrow{\sim} & B^{\max}(s_i\lambda) \\ b & \longmapsto & S_i^* b \end{array} \tag{4.6}$$

*(ii) Let $u_\lambda \in B(U_q(\mathfrak{g})a_\lambda)$ be the corresponding element to $u_\infty \otimes t_\lambda \otimes u_{-\infty}$ through the isomorphism $B(U_q(\mathfrak{g})a_\lambda) \cong B(\infty) \otimes T_\lambda \otimes B(-\infty)$. Then $u_\lambda \in B^{\max}(\lambda)$.*

## 4.3 Criteria for Peter-Weyl type decomposition

For $\lambda \in P$, set

$$\begin{aligned} B(\lambda) &:= \{\tilde{X}_{i_1} \cdots \tilde{X}_{i_k} u_\lambda \,;\, \tilde{X}_i = \tilde{e}_i, \tilde{f}_i,\ i_j \in I\} \setminus \{0\} \subset B(U_q(\mathfrak{g})a_\lambda), \\ B(\lambda)^* &:= \{\tilde{X}_{i_1}^* \cdots \tilde{X}_{i_k}^* u_\lambda^* \,;\, \tilde{X}_i = \tilde{e}_i, \tilde{f}_i,\ i_j \in I\} \setminus \{0\}. \end{aligned}$$

*Remark.*

(i) In terms of crystal graph, $B(\lambda)$ is a connected component of $B(U_q(\mathfrak{g})a_\lambda)$ containing $u_\lambda$.

(ii) If $\lambda$ is a dominant, $B(\lambda)$ coincides with the one in Example 2.4 (iii).

(iii) $B(\lambda)^*$ is stable by the actions of $\tilde{e}_i^*$ and $\tilde{f}_i^*$, that is,

$$\tilde{e}_i^* B(\lambda)^* \subset B(\lambda)^* \sqcup \{0\}, \qquad \tilde{f}_i^* B(\lambda)^* \subset B(\lambda)^* \sqcup \{0\}. \tag{4.7}$$

(iv) By the definition of $\tilde{e}_i^*$ and $\tilde{f}_i^*$, we have

$$\tilde{X}_{i_1}^* \cdots \tilde{X}_{i_k}^* u_\lambda^* = (\tilde{X}_{i_1} \cdots \tilde{X}_{i_k} u_\lambda)^*.$$

Therefore, we get

$$B(\lambda)^* = \{b^* \,;\, b \in B(\lambda)\}. \tag{4.8}$$

We consider the following three conditions.

**(C1)** For any extremal vector $b \in B(\widetilde{U}_q(\mathfrak{g}))$, there exists an embeding of crystal

$$B(wt(b)) \hookrightarrow B(\widetilde{U}_q(\mathfrak{g})),$$

given by $u_{wt(b)} \mapsto b$.

**(C2)** For any $\lambda \in P$, $B(\lambda)_\lambda = \{u_\lambda\}$.

**(C3)** (transitivity of extremal vectors) For any extremal vectors $b_1$, $b_2 \in B(\lambda)$, there exist $i_1, \cdots i_k$ such that

$$b_2 = S_{i_1} \cdots S_{i_k} b_1.$$

**Proposition 4.4** *If $\widetilde{U}_q(\mathfrak{g})$ satisfies the conditions (C1), (C2) and (C3), there exists the following isomorphism of bi-crystal;*

$$B(\widetilde{U}_q(\mathfrak{g})) \cong \bigoplus_{\lambda \in P/W} B^{\max}(\lambda) \otimes B(-\lambda)^*, \tag{4.9}$$

*where $W$ is the Weyl group associated with $\mathfrak{g}$ and an isomorphism of bi-crystal is, by definition, an isomorphism for both crystal structures.*

*Remark.* The tensor product in (4.9) has a different meaning from usual tensor product of crystals. In the tensor product in the R.H.S. of (4.9), $\tilde{e}_i$ and $\tilde{f}_i$ act on the left component and $\tilde{e}_i^*$ and $\tilde{f}_i^*$ act on the right component, that is, for $u \otimes v \in B^{\max}(\lambda) \otimes B(-\lambda)^*$, $\tilde{X}_i(u \otimes v) = \tilde{X}_i u \otimes v$ and $\tilde{X}_i^*(u \otimes v) = u \otimes \tilde{X}_i^* v$ $(X_i = e_i, f_i)$.

*Proof.* In order to show the proposition we shall see the following lemmas.

**Lemma 4.5** *We set*

$$\widetilde{B}(\lambda) := \{\tilde{X}_{i_1}^* \cdots \tilde{X}_{i_k}^* b\,;\, b \in B^{\max}(\lambda),\ i_j \in I,\ X_i = e_i, f_i\} \setminus \{0\}. \tag{4.10}$$

*If the conditions (C1) and (C2) hold, $\widetilde{B}(\lambda)$ has a bi-crystal structure and we have the following isomorphism of bi-crystal;*

$$\begin{aligned} \varphi : B^{\max}(\lambda) \otimes B(-\lambda)^* &\xrightarrow{\sim} \widetilde{B}(\lambda), & (4.11)\\ b \otimes \tilde{X}_{i_1}^* \cdots \tilde{X}_{i_k}^* u_{-\lambda}^* &\mapsto \tilde{X}_{i_1}^* \cdots \tilde{X}_{i_k}^* b. & (4.12) \end{aligned}$$

**Lemma 4.6** *Assume that the conditions (C3) holds. The following (A) and (B) are equivalent:*

**(A)** *$\lambda, \lambda' \in P$ satisfy $\lambda' = w\lambda$ for some $w \in W$,*

**(B)** *$\widetilde{B}(\lambda) = \widetilde{B}(\lambda')$.*

**Lemma 4.7** *We assume that (C3) holds. For $\lambda, \lambda' \in P$ if there is no $w \in W$ such that $\lambda' = w\lambda$, we have*

$$\widetilde{B}(\lambda) \cap \widetilde{B}(\lambda') = \emptyset.$$

*Proof of Lemma 4.5.* To see that $\widetilde{B}(\lambda)$ is a well-defined bi-crystal, it is sufficient to show the stability of $\widetilde{B}(\lambda)$ by the actions of $\tilde{e}_i$, $\tilde{f}_i$, $\tilde{e}_i^*$ and $\tilde{f}_i^*$, which is derived easily from Theorem 4.1 and Lemma 4.2.

Next we shall see the well-definedness of the map $\varphi$. Since the definition of the map $\varphi$ depends on the expression of a right component, we shall show

**(I)** If $X_{i_1}^* \cdots X_{i_k}^* u_{-\lambda}^* = 0$, $X_{i_1}^* \cdots X_{i_k}^* b = 0$ for any $b \in B^{\max}(\lambda)$.

**(II)** If $X_{i_1}^* \cdots X_{i_k}^* u_{-\lambda}^* = X_{j_1}^* \cdots X_{j_l}^* u_{-\lambda}^* \neq 0$, then

$$X_{i_1}^* \cdots X_{i_k}^* b = X_{j_1}^* \cdots X_{j_l}^* b \neq 0$$

for any $b \in B^{\max}(\lambda)$.

(I) Under the assumption $(X_{i_1}^* \cdots X_{i_k}^* u_{-\lambda}^*)^* = X_{i_1} \cdots X_{i_k} u_{-\lambda} = 0$ it is enough to show

$$(X_{i_1}^* \cdots X_{i_k}^* b)^* = X_{i_1} \cdots X_{i_k} b^* = 0.$$

By the definition of $B^{\max}(\lambda)$ and Theorem 3.4 (iii), $b^*$ is an extremal vector and has a weight $-\lambda$. Let $B'$ be the connected component containing $b^*$. Since the condition (C1) holds, $B' \cong B(-\lambda)$ by $b^* \leftrightarrow u_{-\lambda}$. Thus, we obtain (I).
(II) We set $\tilde{Y}_i = \tilde{e}_i$ if $\tilde{X}_i = \tilde{f}_i$ and $\tilde{Y}_i = \tilde{f}_i$ if $\tilde{X}_i = \tilde{e}_i$. Then we have $\tilde{Y}_{i_k}^* \cdots \tilde{Y}_{i_1}^* \tilde{X}_{i_1}^* \cdots \tilde{X}_{i_k}^* b = b$. Therefore, it is sufficent to show that

$$\tilde{Y}_{i_k}^* \cdots \tilde{Y}_{i_1}^* \tilde{X}_{j_1}^* \cdots \tilde{X}_{j_l}^* b = b. \tag{4.13}$$

We know that $b^*$ is an extremal vector and $wt(b^*) = -\lambda$. Let us denote $B'$ for the connected component containing $b^*$. By applying $*$ on the L.H.S of (4.13), we get

$$(\tilde{Y}_{i_k}^* \cdots \tilde{Y}_{i_1}^* \tilde{X}_{j_1}^* \cdots \tilde{X}_{j_l}^* b)^* = \tilde{Y}_{i_k} \cdots \tilde{Y}_{i_1} \tilde{X}_{j_1} \cdots \tilde{X}_{j_l} b^* \in B'$$

Since $wt(X_{i_1}^* \cdots X_{i_k}^* u_{-\lambda}^*) = wt(X_{j_1}^* \cdots X_{j_l}^* u_{-\lambda}^*)$, we get

$$wt(\tilde{Y}_{i_k} \cdots \tilde{Y}_{i_1} \tilde{X}_{j_1} \cdots \tilde{X}_{j_l} b^*) = wt(b^*) = -\lambda. \tag{4.14}$$

By the condition (C2) and the fact $B' \cong B(-\lambda)$, we know that $B'_{-\lambda} = \{b^*\}$. Thus, by (4.14) we have

$$\tilde{Y}_{i_k} \cdots \tilde{Y}_{i_1} \tilde{X}_{j_1} \cdots \tilde{X}_{j_l} b^* = b^*. \tag{4.15}$$

Applying $*$ on the both sides of (4.15), we obtain (4.13).

By Theorem 4.1, it is trivial that

$$\varphi \circ \tilde{X}_i = \tilde{X}_i \circ \varphi \qquad \text{and} \qquad \varphi \circ \tilde{X}_i^* = \tilde{X}_i^* \circ \varphi$$

Thus, the map $\varphi$ is a morphism of bi-crystal.

We shall see the surjectivity of the map $\varphi$. It is enough to show that if $\tilde{X}_{i_1}^* \cdots \tilde{X}_{i_k}^* b \in \widetilde{B}(\lambda)$ for $b \in B^{\max}(\lambda)$, $\tilde{X}_{i_1}^* \cdots \tilde{X}_{i_k}^* u_{-\lambda}^* \neq 0$. It is shown by the fact that $B' \cong B(-\lambda)$, where $B'$ is the connected component including $b^*$.

Finally, we shall see the injectivity of the map $\varphi$. For $u = b_1 \otimes \tilde{X}_{i_1}^* \cdots \tilde{X}_{i_k}^* u_{-\lambda}^*$ and $v = b_2 \otimes \tilde{X}_{j_1}^* \cdots \tilde{X}_{j_l}^* u_{-\lambda}^* \in B^{\max}(\lambda) \otimes B(-\lambda)^*$, we shall prove that if $\varphi(u) = \varphi(v)$, $u = v$. Applying $*$ on $\varphi(u) = \varphi(v)$, we get

$$\tilde{X}_{i_1} \cdots \tilde{X}_{i_k} b_1^* = \tilde{X}_{j_1} \cdots \tilde{X}_{j_l} b_2^*. \tag{4.16}$$

Since both $b_1^*$ and $b_2^*$ are extremal vectors with weight $-\lambda$ and (4.16) implies that $b_1^*$ and $b_2^*$ are contained in the same connected component, by the conditions (C1) and (C2) we obtain $b_1^* = b_2^*$ and $\tilde{X}_{i_1} \cdots \tilde{X}_{i_k} u_{-\lambda}^* = \tilde{X}_{j_1} \cdots \tilde{X}_{j_l} u_{-\lambda}^*$. Thus we get $u = v$. Now we have completed the proof of Lemma 4.5. □

*Proof of Lemma 4.6.* In order to see (A) $\Rightarrow$ (B), it is enough to show that for $\lambda' = s_i \lambda$

$$\widetilde{B}(\lambda') \subset \widetilde{B}(\lambda). \tag{4.17}$$

By Proposition 4.3 (i), $S_i^* : B^{\max}(\lambda) \xrightarrow{\sim} B^{\max}(\lambda')$. Then we have

$$B^{\max}(\lambda') = S_i^* B^{\max}(\lambda) \subset \widetilde{B}(\lambda).$$

Therefore, we get

$$\widetilde{B}(\lambda') \subset \widetilde{B}(\lambda).$$

Since $\lambda = s_i \lambda'$, we also get $\widetilde{B}(\lambda) \subset \widetilde{B}(\lambda')$.

Next we shall see that (B) $\Rightarrow$ (A). For $u_\lambda \in B^{\max}(\lambda)$ there exists $b \in B^{\max}(\lambda')$ and $i_1, \cdots, i_k$ such that

$$u_\lambda = \tilde{X}_{i_1}^* \cdots \tilde{X}_{i_k}^* b.$$

Applying $*$ on the both sides, we get

$$u_\lambda^* = \tilde{X}_{i_1} \cdots \tilde{X}_{i_k} b^*.$$

This implies that $u_\lambda^*$ and $b^*$ belong to the same connected component. Since both $u_\lambda^*$ and $b^*$ are extremal vectors, by (C3) there exist $j_1, \cdots, j_l$ such that

$$u_\lambda^* = S_{j_1} \cdots S_{j_l} b^*. \tag{4.18}$$

Here note that $wt(u_\lambda^*) = -\lambda$ and $wt(b^*) = -\lambda'$ by Theorem 3.4 (iii). Thus, by (3.7) and (4.18) we obtain $\lambda = s_{j_1} \cdots s_{j_l} \lambda'$. □

*Proof of Lemma 4.7* We assume that $\widetilde{B}(\lambda) \cap \widetilde{B}(\lambda') \neq \emptyset$ and $b \in \widetilde{B}(\lambda) \cap \widetilde{B}(\lambda')$. There exist $i_1, \cdots, i_k, j_1, \cdots, j_l \in I$ such that

$$b_1 := \tilde{X}^*_{i_1} \cdots \tilde{X}^*_{i_k} b \in B^{\max}(\lambda),$$
$$b_2 := \tilde{X}^*_{j_1} \cdots \tilde{X}^*_{j_l} b \in B^{\max}(\lambda').$$

By the definition of $B^{\max}(\lambda)$, $b_1^*$ and $b_2^*$ are extremal vectors and

$$wt(b_1^*) = -\lambda \qquad \text{and} \qquad wt(b_2^*) = -\lambda'. \tag{4.19}$$

We also get

$$b_2 = \tilde{X}^*_{j_1} \cdots \tilde{X}^*_{j_l} \tilde{Y}^*_{i_k} \cdots \tilde{Y}^*_{i_1} b_1, \tag{4.20}$$

where $\tilde{Y}_i$ is the same one as in the proof of Lemma 4.5. Applying $^*$ on the both sides of (4.20), we get

$$b_2^* = \tilde{X}_{j_1} \cdots \tilde{X}_{j_l} \tilde{Y}_{i_k} \cdots \tilde{Y}_{i_1} b_1^*.$$

This implies that $b_1^*$ and $b_2^*$ are in the same connected component. By virtue of (C3), there exist $a_1, \cdots, a_n$ such that

$$b_2^* = S_{a_1} \cdots S_{a_n} b_1^*. \tag{4.21}$$

By (3.7), (4.19) and (4.21), we obtain $\lambda' = s_{a_1} \cdots s_{a_n} \lambda$, which is a contradiction. □

*Proof of Proposition 4.4.* By Lemma 4.6, we know that for $\lambda, \lambda' \in P$ if there exists $w \in W$ such that $\lambda' = w\lambda$, $\widetilde{B}(\lambda) = \widetilde{B}(\lambda')$ and otherwise, $\widetilde{B}(\lambda) \cap \widetilde{B}(\lambda') = \emptyset$ by Lemma 4.7. Therefore, we get $\sum_{\lambda \in P} \widetilde{B}(\lambda) = \oplus_{\lambda \in P/W} \widetilde{B}(\lambda)$ and then

$$\bigoplus_{\lambda \in P/W} \widetilde{B}(\lambda) \subset B(\widetilde{U}_q(\mathfrak{g})). \tag{4.22}$$

On the other hand, for any $b \in B(\widetilde{U}_q(\mathfrak{g}))$ let $B'$ be the connected component containing $b^*$. By Theorem 3.8, there exist $\tilde{X}_{i_1}, \cdots, \tilde{X}_{i_k}$ such that $\tilde{X}_{i_1} \cdots \tilde{X}_{i_k} b^*$ is an extremal vector. Then there exists some $\mu \in P$ such that

$$(\tilde{X}_{i_1} \cdots \tilde{X}_{i_k} b^*)^* = \tilde{X}^*_{i_1} \cdots \tilde{X}^*_{i_k} b \in B^{\max}(\mu).$$

This implies $b \in \widetilde{B}(\mu)$. Therefore, we get

$$B(\widetilde{U}_q(\mathfrak{g})) \subset \sum_{\lambda \in P} \widetilde{B}(\lambda) = \bigoplus_{\lambda \in P/W} \widetilde{B}(\lambda). \tag{4.23}$$

By (4.22) and (4.23) we obtain

$$B(\widetilde{U}_q(\mathfrak{g})) = \bigoplus_{\lambda \in P/W} \widetilde{B}(\lambda).$$

Finally, by Lemma 4.5, we get the desired result. □

# 5 Crystallized Peter-Weyl type decomposition for level 0 part of $B(\widetilde{U}_q(\widehat{\mathfrak{sl}_2}))$

In this secton we set $\mathfrak{g} = \widehat{\mathfrak{sl}_2}$. As an application of Proposition 4.4, we shall give the Peter-Weyl type decomposition for the level 0 part of $B(\widetilde{U}_q(\widehat{\mathfrak{sl}_2}))$ explicitly. The notations and terminologies using in this section follow [3], [10, Sec5–Sec7].

## 5.1 Crystals of level 0 part of $\widetilde{U}_q(\widehat{\mathfrak{sl}_2})$

In this subsection we shall review [3],[10].

Let us denote $\widetilde{U}_q(\mathfrak{g})_0$ for the level 0 part of $\widetilde{U}_q(\mathfrak{g})$ given by

$$\widetilde{U}_q(\mathfrak{g})_0 := \{b \in \widetilde{U}_q(\mathfrak{g}) | \langle c, wt(b)\rangle = 0\} = \bigoplus_{\lambda \in P_0 := \{\lambda \in P \mid \langle c,\lambda\rangle = 0\}} U_q(\mathfrak{g}) a_\lambda.$$

We set

$$B_\infty := \{(n) | n \in \mathbf{Z}\}, \qquad (wt(n) = 2n(\Lambda_0 - \Lambda_1)).$$

We define the crystal structure on $B_\infty$ by

$$\begin{aligned} &\tilde{e}_1(n) = \tilde{f}_0(n) = (n-1), && \tilde{e}_0(n) = \tilde{f}_1(n) = (n+1),\\ &\varepsilon_1(n) = \varphi_0(n) = n, && \varepsilon_0(n) = \varphi_1(n) = -n. \end{aligned}$$

*Remark.* We shall identify $B_\infty$ with $\mathbf{Z}$. Then we can consider summation, subtraction and absolute value for elements in $B_\infty$.

We set

$$\mathcal{P}(\infty) := \{(\cdot\cdot, i_k, i_{k+1}, \cdot\cdot, i_{-1}) | i_k \in B_\infty \text{ and if } |k| \gg 0, i_k = (0)\}, \quad (5.1)$$
$$\mathcal{P}(-\infty) := \{(i_0, \cdot\cdot, i_k, i_{k+1}, \cdot\cdot) | i_k \in B_\infty \text{ and if } |k| \gg 0, i_k = (0)\}, \quad (5.2)$$

Then we get the following isomorphisms of crystal ([6],[10, 5.1])

$$B(\infty) \cong \mathcal{P}(\infty), \qquad B(-\infty) \cong \mathcal{P}(-\infty).$$

For an integer $m$ we set

$$\begin{aligned} \mathcal{P}_m := \{p = (\cdots, i_k, i_{k+1}, \cdots, i_{-1}, i_0, i_1, \cdots, i_l, i_{l+1}, \cdots) |\ i_k \in B_\infty \\ \text{if } k \ll 0,\ i_k = (0) \text{ and if } l \gg 0,\ i_{2l} = (m) \text{ and } i_{2l+1} = (-m)\}. \end{aligned} \quad (5.3)$$

Furthermore, let us associate a weight with $p \in \mathcal{P}_m$ by the following:

$$\begin{aligned} wt(p) = (&\sum_{k \in \mathbf{Z}} i_{k-1} + i_k)(\Lambda_0 - \Lambda_1) \\ &+ (l + \textstyle\sum_{k \in \mathbf{Z}} k(\max\{i_{k-1}, -i_k\} - \max\{g_{k-1}, -g_k\}))\delta, \end{aligned} \quad (5.4)$$

where $l$ is an integer and $(g_k)_k$ is the element in $\mathcal{P}_m$ given by $g_k = (0)$ $(k < 0)$ and $g_k = ((-)^k m)$ $(k \geq 0)$. We set $\mathcal{P}_{m,l} = \mathcal{P}_m$ as a set and weight of an element in $\mathcal{P}_{m,l}$ is given by (5.4). For $\lambda = m(\Lambda_0 - \Lambda_1) + l\delta$ we have

$$\mathcal{P}_{m,l} \cong B(U_q(\widehat{\mathfrak{sl}_2})a_\lambda). \tag{5.5}$$

Let us call an element of $\mathcal{P}_{m,l}$ a *path*.

For $n \in \mathbf{Z}_{\geq 0}$, let $\mathcal{P}_m(n)$ be the subset of $\mathcal{P}_m$ with $n$ walls as in [10, Sec 5.3]. For a sequence in $m$-domain configuration (see [10, Sec 7.1]) $\vec{t} = (t_1, \cdots, t_{n-1})$ and a sequence $\vec{c} = (c_1, \cdots, c_{n-1}) \in \mathbf{Z}_{\geq 0}^{n-1}$ let $\mathcal{P}_{m,l}(n; \vec{t}; \vec{c})$ be the same object as in [10, Sec 7.4]. By [10] we get the following

**Theorem 5.1** *$\mathcal{P}_{m,l}(n; \vec{t}; \vec{c})$ is a connected component of $\mathcal{P}_{m,l}$ and any connected component of $\mathcal{P}_{m,l}$ coincides with some $\mathcal{P}_{m,l}(n; \vec{t}; \vec{c})$.*

The crystal $B(\infty)$ has another path realization. (See [3]). For $i \in I$, let $B_i$ be the crystal as in Example 2.4 (ii). We define the map $\Psi_i : B(\infty) \longrightarrow B(\infty) \otimes B_i$ by $\Psi_i(b) = b_0 \otimes \tilde{f}_i^m(0)_i$, where $m = \varepsilon_i^*(b)$ and $b_0 = \tilde{e}_i^{*m} b$. Then we have the following;

**Theorem 5.2 ([3])** *For any $i \in I$, $\Psi_i$ gives a strict embeding of crystals.*

Let us take a sequence $i_1, i_2, \cdots, i_n \in I = \{0, 1\}$ such that $i_k \neq i_{k+1}$.
*Remark.* In the case $\mathfrak{g} = \widehat{\mathfrak{sl}_2}$, there are only two choices of a sequence $i_1, i_2, \cdots$, $(i_1, i_2, \cdots) = (1, 0, 1, 0, \cdots)$ or $(0, 1, 0, 1, \cdots)$.
By iterating $\Psi_i$ we obtain;

$$\Psi_{i_n, \cdots, i_1} : B(\infty) \xrightarrow{\Psi_{i_1}} B(\infty) \otimes B_{i_1} \xrightarrow{\Psi_{i_2} \otimes \mathrm{id}} B(\infty) \otimes B_{i_2} \otimes B_{i_1} \tag{5.6}$$
$$\rightarrow \cdots \rightarrow B(\infty) \otimes B_{i_n} \otimes \cdots B_{i_1}. \tag{5.7}$$

For any $b \in B(\infty)$ if we take $n \gg 0$, $\Psi_{i_n, \cdots, i_1}(b)$ can be written in the following form;

$$\Psi_{i_n, \cdots, i_1}(b) = u_\infty \otimes \tilde{f}_{i_n}^{a_n}(0)_{i_n} \otimes \cdots \otimes \tilde{f}_{i_1}^{a_1}(0)_{i_1} \qquad (a_k \geq 0).$$

For $m > n$, since $\Psi_{i_m, \cdots, i_1}(b) = u_\infty \otimes (0)_{i_m} \otimes \cdots \otimes (0)_{i_{n+1}} \otimes \tilde{f}_{i_n}^{a_n}(0)_{i_n} \otimes \cdots \otimes \tilde{f}_{i_1}^{a_1}(0)_{i_1}$, the sequence $a_1, a_2, \cdots$ does not depend on the choice of $n$. Thus, we have the follwoing embeding of crystal;

$$\Psi : B(\infty) \longrightarrow \{(\cdots, a_n, a_{n-1}, \cdots, a_1) |\ a_n \in \mathbf{Z}_{\geq 0}, \quad a_k = 0 \quad (k \gg 0)\}, \tag{5.8}$$

where $a_k$ means $\tilde{f}_{i_k}^{a_k}(0)_{i_k}$. In [3], the image of the map $\Psi$ is described explicitly for any rank 2 Kac-Moody Lie algebra. In paticular, for $\mathfrak{g} = \widehat{\mathfrak{sl}_2}$ we have

**Proposition 5.3** *We set $z_n = \frac{n}{n-1}$ $(n \geq 2)$. Then we have*

$$\operatorname{Im} \Psi = \left\{ (\cdots, a_n, a_{n-1}, \cdots, a_1) | \quad \begin{array}{l} a_n \in \mathbf{Z}_{\geq 0} \quad \text{and} \quad a_k = 0 \quad (k \gg 0) \\ a_{n+1} \leq z_n a_n \quad (n \geq 2) \end{array} \right\}. \tag{5.9}$$

## 5.2 Description of $B^{\max}(\lambda)$ with level$(\lambda) = 0$

For $b = b_1 \otimes t_\lambda \otimes b_2 \in B(\infty) \otimes T_\lambda \otimes B(-\infty)$, by Theorem 3.4 (iii) we get

$$b^* = b_1^* \otimes t_{-\lambda - wt(b_1) - wt(b_2)} \otimes b_2^* = b_1^* \otimes t_{-wt(b)} \otimes b_2^* \in B(\infty) \otimes T_{-wt(b)} \otimes B(-\infty).$$

Thus, by (4.3) and Theorem 3.2 we can write;

$$B^{\max}(\lambda) = \{b^* | b \in B(\widetilde{U}_q(\mathfrak{g}))_{-\lambda} \text{ and } b \text{ is an extremal vector}\} \tag{5.10}$$

For level 0 weight $\lambda$, we shall describe $B^{\max}(\lambda)$ explicitly. Set $\lambda = m(\Lambda_0 - \Lambda_1) + l\delta$ where $l, m \in \mathbf{Z}$. Let $p \in \mathcal{P}_k$ be an extremal vector. Since all walls (see [10, Sec 5.3]) in $p$ are simultaneously $+$ or $-$ by [10, Theorem 7.18], we have

$$wt(p) = \begin{cases} n(\Lambda_0 - \Lambda_1) + j\delta & \text{if all walls are } +, \\ n(\Lambda_1 - \Lambda_0) + j\delta & \text{if all walls are } -, \end{cases} \tag{5.11}$$

where $n$ is a number of walls in $p$ and $j$ is some integer. Thus, we obtain the following lemma.

**Lemma 5.4** *We set $\lambda = m(\Lambda_0 - \Lambda_1) + l\delta$ with $m \geq 0$ (resp. $m \leq 0$). If $b = (\cdots, i_k, i_{k+1}, \cdots)$ is an extremal vector with the weight $-\lambda$. then $b$ has $|m|$ walls and all walls in $b$ are $-$ (resp. $+$) and $b^* \in \mathcal{P}_{m,l}$, .*

We shall describe the operation $*$ on an extremal vector $p = (\cdots, i_k, i_{k+1}, \cdots) \in B(\widetilde{U}_q(\mathfrak{g})_0)$ more precisely. Set $b = (\cdots, i_k, i_{k+1}, \cdots, i_{-1}) \in \mathcal{P}(\infty) \cong B(\infty)$. We can define walls and domains in $b$ by the similar way to the one in [10, Sec 5.3]

**Definition 5.5** (*i*) *For $b = (\cdots, i_k, i_{k+1}, \cdots, i_{-1}) \in B(\infty)$, by definition, there are $+$ (resp. $-$) walls at position $k \leq -1$ if $i_{k-1} + i_k > 0$ (resp. $i_{k-1} + i_k < 0$) and $|i_{k-1} + i_k|$ is called the number of walls at position $k$. We also call $\sum_{k \leq -1} |i_{k-1} + i_k|$ the total number of walls in $b$.*

(*ii*) *A domain in $b$ is a finite subsequence $i_a, i_{a+1}, \cdots, i_b$ such that $i_{a-1} + i_a \neq 0$, $i_b + i_{b+1} \neq 0$ and $i_j + i_{j+1} = 0$ for $a \leq j < b$, namely, a subsequence surrounded by neighboring two walls or a subsequence $\cdots, i_{c-1}, i_c$ (resp. $i_c, i_{c+1}, \cdots$) such that $i_c + i_{c+1} \neq 0$ and $i_{j-1} + i_j = 0$ for $j \leq c$ (resp. $i_{c-1} + i_c \neq 0$ and $i_{j-1} + i_j = 0$ for $j > c$), namely, a subsequence of $p$ on the left (resp. right) of the left-most (resp. right-most) wall or an empty subsequence without entry which appears between two neighboring walls in the same position.*

*Remark.* The left-most domain is an infinite domain but the right-most domain is not infinite.

**Example 5.6** *For a path $p = (\cdots, 0, 0, 1, -1, 3, -3)$, there is a zero-length domain between entries $-1$ and $3$ and there are one infinite domain $\cdots, 0, 0, 0, 0$ and two finite domains $1, -1$ and $3, -3$.*

If the total number of walls is $n$, there are $n+1$ domains in $b$ and we shall denote $d_0, d_1, \cdots, d_n$ for them, where $d_0$ is the left-most domain and $d_n$ is the right-most domain. All domains except $d_0$ include finite number of entries. For a domain $d_j$ $(j>0)$, let $l_j$ be the length of $d_j$(= number of entries in $d_j$). In particular, the total sum of lenghts

$$l(b) := \sum_{j=1}^{n} l_j$$

is called *length of* $b$.

**Proposition 5.7** *Let* $p = (\cdots, i_k, i_{k+1}, \cdots) \in B(\widetilde{U}_q(\mathfrak{g})_0)$ *be an extremal vector and set* $b = (\cdots, i_k, \cdots, i_{-2}, i_{-1}) \in \mathcal{P}(\infty)$*. Suppose that* $b$ *has* $n$ *walls and set* $l'_j = \sum_{i=1}^{j} l_i$ $(l'_n = l(b))$*. Then we have*

$$b = \underbrace{\tilde{f}^n_{j_{l(b)}} \cdots \tilde{f}^n_{j_{l'_{n-1}+1}}}_{l_n} \cdots\cdots \underbrace{\tilde{f}^2_{j_{l'_2}} \cdots \tilde{f}^2_{j_{l'_1+1}}}_{l_2} \underbrace{\tilde{f}_{j_{l'_1}} \cdots \tilde{f}_{j_1}}_{l_1} u_\infty, \tag{5.12}$$

*with the conditions that* $j_k \neq j_{k+1}$, $\tilde{e}_{j_{k+1}}(\tilde{f}^m_{j_k} \cdots u_\infty) = 0$ *if* $l'_{m-1} < k \leq l'_m$ *and*

$$j_{l(b)} = \begin{cases} 1 & \text{if all walls in } b \text{ are } +, \\ 0 & \text{if all walls in } b \text{ are } -. \end{cases}$$

To show this proposition, we shall see the following lemmas:

**Lemma 5.8** *For* $b = (\cdots, i_k, i_{k+1}, \cdots, i_{-1}) \in \mathcal{P}(\infty)$ *and* $i = 0, 1$ *we set*

$$A^{(i)}_k(b) := \sigma(i)(i_k + 2\sum_{j<k} i_j), \qquad \text{where } \sigma(i) = \begin{cases} + & \text{if } i = 1, \\ - & \text{if } i = 0. \end{cases} \tag{5.13}$$

$(i)$ *If there exists* $k$ *such that*

$$A^{(i)}_k(b) > A^{(i)}_\nu(b) \text{ for } \nu < k \text{ and } A^{(i)}_k(b) \leq A^{(i)}_\nu(b) \text{ for } k < \nu \leq -1,$$

*we have* $\tilde{e}_i b = (\cdots, i_{k-1}, \tilde{e}_i(i_k), i_{k+1}, \cdots, i_{-1})$*, otherwise,* $\tilde{e}_i b = 0$.

$(ii)$ *If there exists* $k$ *such that*

$$A^{(i)}_k(b) \geq A^{(i)}_\nu(b) \text{ for } \nu < k \text{ and } A^{(i)}_k(b) > A^{(i)}_\nu(b) \text{ for } k < \nu \leq -1,$$

*we have* $\tilde{f}_i b = (\cdots, i_{k-1}, \tilde{f}_i(i_k), i_{k+1}, \cdots, i_{-1})$.

$(iii)$ $\varepsilon_i(b) = \max_{k \leq -1}\{A^{(i)}_k(b)\}$.

$(iv)$ *If* $i_k = -i_{k+1}$*, then* $A^{(i)}_k(b) = A^{(i)}_{k+1}(b)$.

(v) *If* $\tilde{e}_i b = (\cdots, i_{k-1}, \tilde{e}_i(i_k), i_{k+1}, \cdots)$, *we have*

$$A_\nu^{(i)}(\tilde{e}_i b) = \begin{cases} A_\nu^{(i)}(b) & \text{if} \quad \nu < k, \\ A_\nu^{(i)}(b) - 1 & \text{if} \quad \nu = k, \\ A_\nu^{(i)}(b) - 2 & \text{if} \quad \nu > k. \end{cases}$$

*Remark.* Since $A_j^{(i)}(b) = A_{j+1}^{(i)}(b)$ for $|j| \gg 0$, there always exists $k$ as in (ii). Thus, $\tilde{f}_i b \neq 0$ for any $i$.

*Proof.* (i) and (ii) are easily obtained by Lemma 3.6 in [3]. Using (2.22) repeatedly, we get (iii). (iv) and (v) are trivial by the definition of $A_k^{(i)}(b)$. $\square$

**Lemma 5.9** *For* $b = (\cdots, i_k, i_{k+1}, \cdots, i_{-1}) \in \mathcal{P}(\infty) \cong B(\infty)$ *suppose that all walls in* $b$ *are* $-$ (*resp.* $+$) *and set* $\tilde{e}_0^{\varepsilon_0(b)} b = (\cdots, i'_k, i'_{k+1}, \cdots, i'_{-1})$. (*resp.* $\tilde{e}_1^{\varepsilon_1(b)} b = (\cdots, i'_k, i'_{k+1}, \cdots, i'_{-1})$). *Then we have*

$$i'_k = -i_{k-1} \qquad \textit{for } k = -1, -2, \cdots \tag{5.14}$$

*Proof.* We shall show the case that all walls are $-$. Let $M := \{m_1, \cdots, m_p\}$ be the set of indices such that $i_{m_j-1} + i_{m_j} < 0$ and $m_1 < m_2 < \cdots < m_p$. Here note that any domain $i_{m_j}, i_{m_j+1}, \cdots, i_{m_{j+1}-1}$ is in the form; $l, -l, \cdots, (-)^{m_{j+1}-m_j-1} l$. By this fact and Lemma 5.8 (iii), we have

$$A_{m_1}^{(0)}(b) < A_{m_2}^{(0)}(b) < \cdots < A_{m_p}^{(0)}(b) = \varepsilon_0(b), \tag{5.15}$$

$$A_{m_j}^{(0)}(b) - A_{m_{j-1}}^{(0)}(b) = -i_{m_j-1} - i_{m_j} > 0. \tag{5.16}$$

We set $n_j := -i_{m_j-1} - i_{m_j}$. Here note that by Lemma 5.8 (i) and (iv), $\tilde{e}_i$ acts on some $i_k$ ($k \in M$) or $\tilde{e}_i b = 0$. In this case, by (5.15) we know that $\tilde{e}_0$ acts on $i_{m_p}$. Since $\tilde{e}_0(i_k) = i_k + 1$, if we apply $\tilde{e}_0^{n_p}$ on $b$, by Lemma 5.8 (v) we get

$$\begin{aligned} \tilde{e}_0^{n_p} b &= (\cdots, i_{m_p-1}, \tilde{e}_0^{n_p}(i_{m_p}), i_{m_p+1}, \cdots) && (5.17) \\ &= (\cdots, i_{m_p-1}, -i_{m_p-1}, i_{m_p+1}, \cdots) && (5.18) \end{aligned}$$

Furthermore, if we apply $\tilde{e}_0$ on $\tilde{e}_0^{n_p} b$, it acts on $i_{m_{p-1}}$ by Lemma 5.8 (ii) (v). Repeating this, we obtain

$$\tilde{e}_0^{\varepsilon_0(b)} b = (\cdots, \tilde{e}_0^{n_1}(i_{m_1}), \cdots, \tilde{e}_0^{n_j}(i_{m_j}), \cdots, \tilde{e}_0^{n_p}(i_{m_p}), \cdots).$$

Here note that

$$\sum_{j=1}^{p} n_j = -\sum_{j=1}^{p} (i_{m_j-1} + i_{m_j}) = A_{m_p}^{(0)}(b) = \varepsilon_0(b).$$

Since $i_{m_j-1} + \tilde{e}_0^{n_j}(i_{m_j}) = 0$, we have

$$i'_{m_j} = -i_{m_j-1}. \tag{5.19}$$

If $k \notin M$, then $i'_k = i_k$ and $i_{k-1} + i_k = 0$. This implies

$$i'_k = -i_{k-1} \qquad \text{for} \quad k \notin M. \tag{5.20}$$

By (5.19) and (5.20) we obtained the desired result. The case that all walls are $+$ is also shown similarly. $\square$

*Proof of Proposition 5.7.* We shall show by the induction on $l(b)$. If $l(b) = 1$, $b$ is in the following form;

$$b = (\cdots, 0, 0, 0, k).$$

If $k > 0$, all walls in $b$ are $+$ and $b = \tilde{f}_1^k u_\infty$. If $k < 0$, all walls in $b$ are $-$ and $b = \tilde{f}_0^{|k|} u_\infty$. Suppose that $l(b) > 1$ and all walls in $b$ are $-$. Lemma 5.9 implies that all walls in $\tilde{e}_0^{\varepsilon_0(b)} b$ are $+$ and $l(\tilde{e}_0^{\varepsilon_0(b)} b) = l(b) - 1$. Then, by the hypothesis of the induction,

$$\tilde{e}_0^{\varepsilon_0(b)} b = \underbrace{\tilde{f}_1^n \cdots \tilde{f}_{j_{l'_{n-1}+1}}^n}_{-1+l_n} \cdots\cdots \underbrace{\tilde{f}_{j_{l'_2}}^2 \cdots \tilde{f}_{j_{l'_1+1}}^2}_{l_2} \underbrace{\tilde{f}_{j_{l'_1}} \cdots \tilde{f}_{j_1}}_{l_1} u_\infty. \tag{5.21}$$

Since $\varepsilon_0(b) = n$ and $b = \tilde{f}_0^n \tilde{e}_0^n b$, we obtain the desired result. The case that all walls in $b$ are $+$ is also shown similarly. $\square$

We shall introduce the following lemma similar to Lemma 5.8.

**Lemma 5.10** *For $b = (\cdots, a_k, a_{k-1}, \cdots, a_2, a_1) \in B_I$ we set*

$$\widehat{A}_k^{(0)}(b) \;:=\; a_{2k-1} + 2\sum_{j>k}(a_{2j-1} - a_{2j-2}) \quad (k \geq 1), \tag{5.22}$$

$$\widehat{A}_k^{(1)}(b) \;:=\; a_{2k} + 2\sum_{j>k}(a_{2j} - a_{2j-1}) \quad (k \geq 1). \tag{5.23}$$

(i) *If there exists $k$ such that $\widehat{A}_k^{(i)}(b) > \widehat{A}_\nu^{(i)}(b)$ for $k < \nu$ and $\widehat{A}_k^{(i)}(b) \geq \widehat{A}_\nu^{(i)}(b)$ for $k > \nu \geq 1$,*

$$\tilde{e}_i b = \begin{cases} (\cdots, a_{2k-1} - 1, \cdots) & \text{if } i = 0, \\ (\cdots, a_{2k} - 1, \cdots) & \text{if } i = 1, \end{cases}$$

*otherwise, $\tilde{e}_i b = 0$.*

*If there exists $k$ such that $\widehat{A}_k^{(i)}(b) \geq \widehat{A}_\nu^{(i)}(b)$ for $k < \nu$ and $\widehat{A}_k^{(i)}(b) > \widehat{A}_\nu^{(i)}(b)$ for $k > \nu \geq 1$,*

$$\tilde{f}_i b = \begin{cases} (\cdots, a_{2k-1} + 1, \cdots) & \text{if } i = 0, \\ (\cdots, a_{2k} + 1, \cdots) & \text{if } i = 1, \end{cases}$$

(ii) $\varepsilon_i(b) = \max_{k \geq 1}\{\widehat{A}_k^{(i)}\}$.

It is easy to show this lemma by Lemma 1.3.6 in [3].

We set

$$B_I \;:=\; \left\{ (\cdot\cdot, a_k, a_{k-1}, \cdots, a_1) \middle| \begin{array}{l} a_k \in \mathbf{Z}_{\geq 0},\ a_{k+1} \leq a_k \\ a_k = 0 \quad \text{for} \quad k \gg 0 \end{array} \right\} \subset \Psi(B(\infty)), \tag{5.24}$$

$$B_{II} \;:=\; \left\{ (\cdots, i_k, i_{k+1}, \cdots, i_{-1}) \in \mathcal{P}(\infty) \,\middle|\, \begin{array}{l} i_{2k} \geq 0, i_{2k-1} \leq 0, \\ |i_{k-1}| \leq |i_k|. \end{array} \right\}. \tag{5.25}$$

where we set $(\cdots, a_k, \cdots, a_2, a_1) := u_\infty \otimes \cdots \otimes \tilde{f}_1^{a_2}(0)_1 \otimes \tilde{f}_0^{a_1}(0)_0$.
The following lemma guarantees that $B_I$ and $B_{II}$ are stable by the action of $\tilde{e}_i$.

**Lemma 5.11** *(i) If $b = (\cdots, a_k, a_{k-1}, \cdots, a_1) \in B_I$ and $\tilde{e}_i b \neq 0$, $\tilde{e}_i b \in B_I$.*

*(ii) If $b = (\cdots, i_k, i_{k+1}, \cdots, i_{-1}) \in B_{II}$ and $\tilde{e}_i b \neq 0$, $\tilde{e}_i b \in B_{II}$.*

*Proof.* (i) We consider the case of $i = 0$ and $b \in B_I$. We assume that $a_{2k+1}$ changes to $a_{2k+1} - 1$ by the action of $\tilde{e}_0$ and $a_{2k+1} = a_{2k+2}$. $\widehat{A}_k^{(0)}(b) - \widehat{A}_{k+1}^{(0)}(b) = a_{2k+1} - 2a_{2k+2} + a_{2k+3} = a_{2k+3} - a_{2k+2} \leq 0$. This means $\widehat{A}_k^{(0)}(b) \leq \widehat{A}_{k+1}^{(0)}(b)$, which contradicts Lemma 5.10. Thus if $a_{2k+1}$ changes to $a_{2k+1} - 1$ by the action of $\tilde{e}_0$, $a_{2k+2} < a_{2k+1}$. As for the $i = 1$-case, we can show similarly.
(ii) For $b = (\cdots, i_k, i_{k+1}, \cdots, i_{-1}) \in B_{II}$ suppose that $\tilde{e}_0 b = (\cdots, i_k + 1, \cdots) \neq 0$. Let $A_k^{(i)}(b)$ be as in Lemma 5.8. By the assumption we have $A_k^{(0)}(b) > A_{k-1}^{(0)}(b)$ and then $i_k + i_{k-1} < 0$. Suppose that $i_k \geq 0$. Then $-i_{k-1} > i_k \geq 0$. This means $|i_k| < |i_{k-1}|$, which is a contradiction. Thus we have $i_k < 0$. This implies $0 \leq i_{k-1} < -i_k$ and then $0 \leq i_{k-1} \leq -i_k - 1$. Therefore, we get $|i_{k-1}| \leq |i_k + 1| = |\tilde{e}_0(i_k)|$. Then it follows that if $\tilde{e}_0 b \neq 0$, $\tilde{e}_0 b \in B_{II}$. The $i = 1$-case can be shown by the similar way. $\square$

*Remark.*

(*i*) We know that a vector in $B_I$ is in the image of $\Psi$ by Proposition 5.3.

(*ii*) Any element $b = (\cdots, a_2, a_1) \in B_I$ is in the form:

$$b = (\cdots, 0, 0, \underbrace{1 \cdots, 1}_{l_1}, \underbrace{2, \cdots, 2}_{l_2}, \cdots, \underbrace{n, \cdots, n}_{l_n}).$$

Now, we shall show the following proposition:

**Proposition 5.12** *For $b \in B(\infty)$, suppose that*

$$\Psi(b) = (\cdots 0, 0, d_1, \cdots, d_n) \in B_I, \tag{5.26}$$

*where as in Remark (ii) as above;*

$$d_m = \underbrace{m, \cdots, m}_{l_m} = \underbrace{\tilde{f}_{s_1^m}^m \otimes \cdots \otimes \tilde{f}_{s_{l_m}^m}^m}_{l_m} \in B_{s_1^m} \otimes \cdots \otimes B_{s_{l_m}^m}, \quad (s_j^m = 0, 1).$$

*Then we have*

$$\Phi(b) = (\cdots 0, 0, \hat{d}_1, \cdots, \hat{d}_n) \in B_{II}, \tag{5.27}$$

*where* $\Phi$ *is the isomorphism* $\Phi : B(\infty) \xrightarrow{\sim} \mathcal{P}(\infty)$ *and*

$$\hat{d}_m = \underbrace{\sigma(s_1^m)m, \cdots, \sigma(s_{l_m}^m)m}_{l_m} \in B_\infty{}^{\otimes l_m},$$

To show the proposition, we shall prepare several lemmas.

**Lemma 5.13** *Let* $A_k^{(i)}(b)$ *be as in Lemma 5.8 and* $\widehat{A}_k^{(i)}(b)$ *be as in Lemma 5.10. Let* $b_1 = (\cdots, a_2, a_1) \in B_I$ *be as in (5.26) and* $b_2 = (\cdots, i_{-2}, i_{-1}) \in B_{II}$ *be as in (5.27). Then* $\tilde{f}_i b_1 = (\cdots, a_k + 1, \cdots)$ *(resp.* $\tilde{e}_i b_1 = (\cdots, a_k - 1, \cdots)$*) if and only if* $\tilde{f}_i b_2 = (\cdots, \tilde{f}_i(i_{-k}), \cdots)$ *(resp.* $\tilde{e}_i b_2 = (\cdots, \tilde{e}_i(i_{-k}), \cdots)$*).*

*Proof.* First, we shall show

$$\widehat{A}_k^{(i)}(b_1) = A_{-2k+1-i}^{(i)}(b_2), \tag{5.28}$$

for any $k \in \mathbf{Z}_{>0}$ and $i \in \{0, 1\}$. For this purpose, we shall see the following;

**Lemma 5.14** **(i)** *If* $a_{2k-1+i}$ *is in* $d_m$ *(i.e.* $a_{2k-1+i} = m$*),*

$$\widehat{A}_k^{(i)}(b_1) = \sharp\{j | s_1^j = i,\ 1 \le j \le m\} - \sharp\{j | s_1^j = 1 - i,\ 1 \le j \le m\}. \tag{5.29}$$

**(ii)** *If* $i_{-2k+1-i}$ *is in* $\hat{d}_m$*,*

$$A_{-2k+1-i}^{(i)}(b_2) = \sharp\{j | s_1^j = i,\ 1 \le j \le m\} - \sharp\{j | s_1^j = 1 - i,\ 1 \le j \le m\}. \tag{5.30}$$

*Remark.* Even if $l_m = 0$, $s_1^m$ and $s_{l_m}^m$ are uniquely determined by applying the conditions $s_{l_m}^m \ne s_1^{m+1}$, $s_1^m \ne s_{l_{m-1}}^{m-1}$ and if $l_m$ is even, $s_1^m \ne s_{l_m}^m$ to the nearest non-empty domains. That is, let $d_a, d_b, d_c$ $(a < b < c)$ be domains such that $d_a$ and $d_c$ are non-empty and $d_b$ is the empty domain surrounded by $d_a$ and $d_c$. Then $s_1^b$ and $s_{l_b}^b$ are given by $s_1^b = 1 - s_{l_a}^a$ and $s_{l_b}^b = 1 - s_1^c$.

*Proof.* (i) We shall consider the $i = 0$ case. We know that $a_{2k-2}$ is the left-most entry of some $d_m$ if and only if $a_{2k-1} < a_{2k-2}$. Now, $a_{2k-2}$ implies $\tilde{f}_1^{a_{2k-2}}(0)_1$. Since the index of the left-most entry of $d_j$ is $s_1^j$, if $a_{2k-1}$ is in some $d_m$ (*i.e.* $a_{2k-1} = m$), by the remark as above and (5.22), we have

$$\begin{aligned}
\widehat{A}_k^{(0)}(b_1) &= a_{2k-1} - 2\sharp\{j | s_1^j = 1,\ \ 1 \le j \le m\} \\
&= m - 2\sharp\{j | s_1^j = 1,\ 1 \le j \le m\} \\
&= \sharp\{j | s_1^j = 0,\ 1 \le j \le m\} + \sharp\{j | s_1^j = 1,\ 1 \le j \le m\} - 2\sharp\{j | s_1^j = 1,\ 1 \le j \le m\} \\
&= \sharp\{j | s_1^j = 0,\ 1 \le j \le m\} - \sharp\{j | s_1^j = 1,\ 1 \le j \le m\}
\end{aligned}$$

$\widehat{A}^{(1)}_k(b_1)$ is also given similarly.
(ii) We shall consider the $i=0$ case. By (5.13), we can write

$$A^{(0)}_k(b_2) = -\sum_{j\le k}(i_{j-1}+i_j). \tag{5.31}$$

We know that $i_{j-1}+i_j \neq 0$ if and only if $i_j$ is the left-most element of some $\hat{d}_m$. Here note that if $j$ is odd, $i_{j-1}+i_j \le 0$ and if $j$ is even, $i_{j-1}+i_j \ge 0$ by the conditions of $B_{II}$. Let $i_j$ be the left-most element of $\hat{d}_m$. If $j$ is even, $s^m_1=1$ and if $j$ is odd, $s^m_1=0$. Thus, if $i_{-2k+1}$ is in $\hat{d}_m$, by the remark as above and (5.31) we have

$$\begin{aligned} A^{(0)}_{-2k+1}(b_2) &= -(\sharp\{j|s^j_1=1,\ 1\le j\le m\}-\sharp\{j|s^j_1=0,\ 1\le j\le m\}) & (5.32)\\ &= \sharp\{j|s^j_1=0,\ 1\le j\le m\}-\sharp\{j|s^j_1=1,\ 1\le j\le m\}. & (5.33)\end{aligned}$$

$A^{(1)}_{-2k}(b_2)$ is also obtained by the simlar argument. □

*Proof of lemma 5.13.* By Lemma 5.14 we obtain (5.28).

For $i=0$ case, we set $\tilde{f}_0(b_1)=(\cdots,a_{2k-1}+1,\cdots)$. By Lemma 5.10 this implies that $\widehat{A}^{(0)}_k(b_1)\ge \widehat{A}^{(0)}_\nu(b_1)$ for $k<\nu$ and $\widehat{A}^{(0)}_k(b_1) > \widehat{A}^{(0)}_\nu(b_1)$ for $k>\nu\ge 1$. Thus, by (5.28) we have $A^{(0)}_{-2k+1}(b_2)\ge A^{(0)}_{-2\nu+1}(b_2)$ for $k<\nu$ and $A^{(0)}_{-2k+1}(b_2) > A^{(0)}_{-2\nu+1}(b_2)$ for $k>\nu\ge 1$. Furthermore, by simple calculations and the definition of $B_{II}$, for any $j\in \mathbf{Z}_{>0}$ we get

$$A^{(0)}_{-2j}(b_2)-A^{(0)}_{-2j+1}(b_2) = i_{-2j+1}+i_{-2j}\le 0, \tag{5.34}$$

$$A^{(0)}_{-2j-1}(b_2)-A^{(0)}_{-2j}(b_2) = i_{-2j-1}+i_{-2j}\ge 0. \tag{5.35}$$

This implies that

$$A^{(0)}_{-2j-1}(b_2)\ge A^{(0)}_{-2j}(b_2)\le A^{(0)}_{-2j+1}(b_2).$$

Then by Lemma 5.8 (ii), we know that the action by $\tilde{f}_0$ on $b_2$ never touch $i_{-2j}$. Then we get $\tilde{f}_0 b_2=(\cdots,\tilde{f}_0(i_{-2k+1}),\cdots)$. Arguing similarly, we have that $\tilde{f}_0 b_1=(\cdots,a_{2k+1}+1,\cdots)$ if $\tilde{f}_0 b_2=(\cdots,\tilde{f}_0(i_{-2k+1}),\cdots)$. The other cases are shown similarly. □

**Lemma 5.15** *For $b\in B_I$ (resp. $b\in B_{II}$) there exists $i\in\{0,1\}$ such that*

$$l(\tilde{e}_i^{\varepsilon_i(b)}(b)) = l(b)-1, \tag{5.36}$$

*where $l(b)$ is a length of $b$ given as the largest number $k>0$ such that $a_k\neq 0$ for $b=(\cdots,a_j,a_{j-1},\cdots,a_1)$ (resp. $i_{-k}\neq(0)$ for $b=(\cdots,i_{-k},i_{-k+1},\cdots,i_{-1})$).*

*Proof of Lemma 5.15.* If $l(b)$ is even, we choose $i=1$ and if $l(b)$ is odd, we choose $i=0$. (Here note that we chose the sequence of indices :$(\cdots,1,0,1,0)$ for $B_I$.). We assume $l(b)$ is odd and then $i=0$, and set $l(b)=2k-1$ and

$$\tilde{e}_0^{\varepsilon_0(b)}b=(\cdots,b_{2k},b_{2k-1},\cdots,b_1)\quad(\text{resp. }(\cdots,j_{-2k},j_{-2k+1},\cdots,j_{-1})).$$

Suppose that $b_{2k-1}>0$ (resp. $j_{-2k+1}>0$). Then we have $\widehat{A}_k^{(0)}(\tilde{e}_0^{\varepsilon_0(b)}b)=b_{2k-1}>0$ (resp. $A_{-2k+1}^{(0)}(\tilde{e}_0^{\varepsilon_0(b)}b)=j_{-2k+1}>0$) and then by Lemma 5.10 (ii) (resp. Lemma 5.8(iii)) $\varepsilon_0(\tilde{e}_0^{\varepsilon_0(b)}b)>0$, which is a contradiction (see Example 2.4 (iv)). Thus, we get $b_{2k-1}=0$ (resp. $j_{-2k+1}=0$). For $b\in B_I$, $a_{2j}$ implies $\tilde{f}_1^{a_{2j}}(0)_1$. Then $\tilde{e}_0$ never touch $a_{2k-2}$. Then we have $a_{2k-2}=b_{2k-2}$. By the definition of $B_I$, we know that $b_{2k-2}=a_{2k-2}\geq a_{2k-1}>0$. Now, we obtain (5.36) for $b\in B_I$.

For $b\in B_{II}$, by the assumption $l(b)$ is odd and by the definition of $B_{II}$, we have $i_{-2k+1}+i_{-2k+2}\geq 0$. Then, by the facts that $\varphi_0(i_{-2k+1})=i_{-2k+1}$ and $\varepsilon_0(i_{-2k+2})=-i_{-2k+2}$, we get $\varphi_0(i_{-2k+1})\geq\varepsilon_0(i_{-2k+2})$ and in general, $\varphi_0(\tilde{e}_0^m(i_{-2k+1}))=\varphi_0(i_{-2k+1})+m\geq\varepsilon_0(i_{-2k+2})=-i_{-2k+2}$ for $m\geq 0$. Thus, by (2.24) we obtain

$$\tilde{e}_0^m(i_{-2k+1}\otimes i_{-2k+2})=\tilde{e}_0^m(i_{-2k+1})\otimes i_{-2k+2}.$$

This implies that action of $\tilde{e}_0^m$ $(0\leq m\leq\varepsilon_0(b))$ never touch the entry $i_{-2k+2}$ and then $i_{-2k+2}=j_{-2k+2}\neq 0$. Now, we get (5.36) for $b\in B_{II}$. □

*Proof of Proposition 5.12.* Let us complete the proof of Proposition 5.12 by using the induction on $l(\Psi(b))$. For the case $l(\Psi(b))=1$, we can write $\Psi(b)=(\cdots,0,0,a)=u_\infty\otimes\tilde{f}_0^a(0)_0$ $(a>0)$. Then we obtain $\Phi(b)=\tilde{f}_0^a(\cdots,0,0,0)=(\cdots,0,0,-a)\in B_{II}$. We assume $l(\Psi(b))>1$. Set $\Psi(b)=(\cdots,a_k,a_{k-1},\cdots,a_1)$ and $\Phi(b)=(\cdots,i_k,i_{k+1},\cdots,i_{-1})$. By Lemma 5.15, there exists $i$ such that $l(\tilde{e}_i^{\varepsilon_i(b)}\Psi(b))=l(\Psi(b))-1$. Since $B_I$ is stable by the action of $\tilde{e}_i$ by Lemma 5.11, we can set $\tilde{e}_i^{\varepsilon_i(b)}\Psi(b)=(\cdots,a'_k,\cdots,a'_2,a'_1)=(\cdots,0,0,d'_1,d'_2,\cdots,d'_r)\in B_I$ with

$$d'_m=\underbrace{m,\cdots,m}_{l'_m}=\tilde{f}_{t_1^m}^m\otimes\cdots\otimes\tilde{f}_{t_{l'_m}^m}^m,$$

where $t_j^m\in\{0,1\}$ and $l'_m$ is the length of $d'_m$. By the hypothesis of the induction, we can write $\tilde{e}_i^{\varepsilon_i(b)}(\Phi(b))=\Phi(\tilde{e}_i^{\varepsilon_i(b)}b)=(\cdots,i'_{-k},\cdots,i'_{-2},i'_{-1})=(\cdots 0,0,\hat{d}'_1,\hat{d}'_2,\cdots,\hat{d}'_r)$ with

$$\hat{d}'_m=\sigma(t_1^m)m,\cdots,\sigma(t_{l'_m}^m)m.$$

We consider $i=0$ case. If $\tilde{f}_0\tilde{e}_0^{\varepsilon_0(b)}\Psi(b)=(\cdots,a'_{2k+1}+1,\cdots)$, by Lemma 5.13 we get $\tilde{f}_0\tilde{e}_0^{\varepsilon_0(b)}\Phi(b)=(\cdots,\tilde{f}_0(i'_{-2k-1}),\cdots)$. Here note that $\tilde{f}_0(i'_{-2k-1})=$

$i'_{-2k-1}-1$ and $i'_{-2k-1}\le 0$ by the definition of $B_{II}$, then $|\tilde f_0(i'_{-2k-1})|=|i'_{-2k-1}|+1=a'_{2k+1}+1$. Repeating this $\varepsilon_0(b)$ times, since $\tilde f_0^{\varepsilon_0(b)}\tilde e_0^{\varepsilon_0(b)}\Phi(b)=\Phi(b)$ and $\tilde f_0^m\tilde e_0^{\varepsilon_0(b)}\Psi(b)\in B_I$ $(0\le m\le\varepsilon_0(b))$, we obtain the desired result. $\square$

**Proposition 5.16** *For $b\in B(\infty)$ suppose that all walls in $\Phi(b)=(\cdots,i_k,i_{k+1},\cdots,i_{-1})\in\mathcal{P}(\infty)$ are $-$ (resp. $+$) and the total number of walls is $n\in\mathbf{Z}_{>0}$. (Then there are $n$ finite domains.). Let $l_j$ $(1\le j\le n)$ be the length of the $j$-th finite domain. Then we have*

$$\Phi(b^*)=(\cdots,0,0,\bar d_1,\bar d_2,\cdots,\bar d_n),$$

*with*

$$\bar d_j=\underbrace{(-)^{k_j}j,(-)^{k_j+1}j,\cdots,(-)^{k_j+l_j-1}j}_{l_j},\tag{5.37}$$

$$(\text{resp. } \bar d_j=\underbrace{(-)^{k_j+1}j,(-)^{k_j+2}j,\cdots,(-)^{k_j+l_j}j}_{l_j}),\tag{5.38}$$

*where $k_j\in\{-1,-2,\cdots\}$ is the position of the left-most entry of $\bar d_j$.*

*Proof.* We assume that all walls in $b$ are $-$. Here note the following: if $b\in B(\infty)$ can be written $b=\tilde f_i^m b'$ with $\tilde e_i b'=0$ (that is, $\varepsilon_i(b)=m$),

$$\Psi_i(b^*)=b'\otimes\tilde f_i^m(0)_i\tag{5.39}$$

since $\varepsilon_i^*(b^*)=\varepsilon_i(b)=m$. (See 5.1.). By Proposition 5.7, we can write $b$ in the following form;

$$b=\underbrace{\tilde f_0^n\cdots\tilde f^n_{j_{l'_{n-1}+1}}}_{l_n}\cdots\cdots\underbrace{\tilde f^2_{j_{l'_2}}\cdots\tilde f^2_{j_{l'_1+1}}}_{l_2}\underbrace{\tilde f_{j_{l'_1}}\cdots\tilde f_{j_1}}_{l_1}u_\infty,\quad (l'_m=\sum_{j=1}^m l_j)$$

with the conditions $j_k\ne j_{k+1}$ $(1\le k<l'_n)$ and

$$\tilde e_{j_{k+1}}(\tilde f^m_{j_k}\cdots u_\infty)=0,\tag{5.40}$$

for $l'_{m-1}<k\le l'_m$ and $1\le m\le n$. By virtue of (5.40), we can apply (5.39) to $b^*$ repeatedly and obtain;

$$\Psi(b^*)=(\cdots,0,0,d_1,d_2,\cdots,d_n),\tag{5.41}$$

where

$$d_m=\underbrace{m,\cdots,m}_{l_m}=\underbrace{\tilde f^m_{s_1^m}\otimes\cdots\otimes\tilde f^m_{s^m_{l_m}}}_{l_m}\quad\text{and}\quad s^n_{l_n}=0\quad(1\le m\le n).$$

Since the vector (5.41) is in $B_I$, by Proposition 5.12 we have

$$\Phi(b^*) = (\cdots, 0, 0, \hat{d}_1, \hat{d}_2, \cdots, \hat{d}_n),$$

with

$$\hat{d}_m = \underbrace{\sigma(s_1^m)m, \cdots, \sigma(s_{l_m}^m)m}_{l_m} \in B_\infty^{\otimes l_m}.$$

We set $\Phi(b^*) = (\cdots, i_{-k}, \cdots, i_{-1})$ and $\Psi(b^*) = (\cdots, a_k, \cdots, a_1)$. If $k$ is odd (resp. even), we have $a_k = \tilde{f}_0^{a_k}(0)_0$ (resp. $a_k = \tilde{f}_1^{a_k}(0)_1$) and $i_{-k} < 0$ (resp. $i_{-k} > 0$). Then we have $\hat{d}_j = \bar{d}_j$ $(1 \le j \le n)$. $\square$

Now, we shall give the similar description for $B(-\infty)$. By the definition of $\wedge$, we have

$$\begin{array}{ccc} B_\infty^\wedge & \xrightarrow{\sim} & B_\infty \\ (n)^\wedge & \mapsto & (-n). \end{array}$$

As mentioned in Example 2.4 (iv), $B(\mp\infty) = B(\pm\infty)^\vee \cong B(\pm\infty)^\wedge$. Then we can identify $\vee$ with $\wedge$ on $B(\pm\infty)$. We obtain the following isomorphism:

$$\Phi^+ := \wedge \circ \Phi \circ \wedge : B(-\infty) \xrightarrow{\sim} \mathcal{P}(-\infty).$$

*Remark.* For $b \in B(\infty)$, if we set

$$\Phi(b) = (\cdots, i_{-k}, i_{-k+1}, \cdots, i_{-1}),$$
$$\Phi^+(b^\wedge) = (j_0, \cdots, j_k, j_{k+1}, \cdots),$$

we get $j_k = -i_{-k-1}$.

**Proposition 5.17** *For $b \in B(-\infty)$ suppose that all walls in $\Phi^+(b) = (i_0, \cdots, i_k, i_{k+1}, \cdots) \in \mathcal{P}(-\infty)$ are $-$ (resp. $+$) and the number of walls is $n \in \mathbf{Z}_{>0}$. (Then there are $n$ finite domains.). Let $l_j$ $(1 \le j \le n)$ be the length of the finite domain $\bar{d}_j$. Then we have*

$$\Phi^+(b^*) = (\bar{d}_n, \bar{d}_{n-1}, \cdots, \bar{d}_1, 0, 0, \cdots)$$

*where*

$$\bar{d}_j := \underbrace{(-)^{k_j+1}j, (-)^{k_j+2}j, \cdots, (-)^{k_j+l_j}j}_{l_j}, \tag{5.42}$$

$$\text{(resp.} \qquad \bar{d}_j := \underbrace{(-)^{k_j}j, (-)^{k_j+1}j, \cdots, (-)^{k_j+l_j-1}j}_{l_j}), \tag{5.43}$$

*where $k_j \in \{0, 1, 2, \cdots\}$ is the position of the left-most entry of $\bar{d}_j$.*

*Proof.* If all the walls in some element $b_0 \in B(\infty)$ are $+$ (resp. $-$), by the Remark as above all the walls in $b := b_0^\wedge = b_0^\vee$ are $-$ (resp. $+$). By the commutativity of $\vee$ and $*$ and Proposition 5.16, we get the desired result. $\square$

For $m \in \mathbf{Z}$, we set

$$\mathcal{P}_m(-\infty) = \left\{ (i_0, i_1, \cdots, i_k, \cdots) \left| \begin{array}{l} i_k \in B_\infty \quad \text{and if} \quad |k| \gg 0, \\ i_{-k} = (0), \quad i_{2k} = (m), \quad i_{2k+1} = (-m) \end{array} \right. \right\}.$$

For $\lambda = m(\Lambda_0 - \Lambda_1) + l\delta$ by [10, 5.2], there exists the isomorphism

$$\begin{array}{rcl} T_\lambda \otimes \mathcal{P}(-\infty) & \cong & \mathcal{P}_m(-\infty) \\ t_\lambda \otimes (i_0, \cdots, i_{2k}, i_{2k+1}, \cdots) & \leftrightarrow & (i_0 + m, \cdots, i_k + (-)^k m, \cdots) \end{array} \tag{5.44}$$

where weight of $b$ in $\mathcal{P}_m(-\infty)$ is given by $wt(b) = \lambda + wt(b')$ $(t_\lambda \otimes b' \leftrightarrow b)$.
Now, we can describe the operation $*$ on extremal vectors. For a level 0 weight $\lambda = m(\Lambda_0 - \Lambda_1) + l\delta$, let $b = (\cdots, i_k, i_{k+1}, \cdots) \in B(\widetilde{U}_q(\mathfrak{g})_0)_{-\lambda}$ be an extremal vector. By (5.11) and Lemma 5.4, if $m > 0$ (resp. $m < 0$), then all walls in $b$ are $-$ (resp. $+$) and the number of walls is $|m|$. There are $|m| - 1$ domains in $b$ denoted $d_1, d_2, \cdots, d_{|m|-1}$. Let $l_j$ be the length of $d_j$ and $k_j$ be the position of the left-most entry in a domain $d_j$ with $l_j > 0$.

**Theorem 5.18** *For $\lambda = m(\Lambda_0 - \Lambda_1) + l\delta$ with $m > 0$ (resp. $m < 0$) let $b = (\cdots, i_k, i_{k+1}, \cdots) \in B(\widetilde{U}_q(\mathfrak{g})_0)_{-\lambda}$ be an extremal vector as above. Set*

$$\bar{d}_j := i'_{k_j}, i'_{k_j+1}, \cdots, i'_{k_{j+1}-1} = \underbrace{(-)^{k_j} j, (-)^{k_j+1} j, \cdots, (-)^{k_{j+1}-1} j}_{l_j} \tag{5.45}$$

$$(\text{resp.} \quad \bar{d}_j := i'_{k_j}, i'_{k_j+1}, \cdots, i'_{k_{j+1}-1} = \underbrace{(-)^{k_j+1} j, (-)^{k_j+2} j, \cdots, (-)^{k_{j+1}} j}_{l_j}). \tag{5.46}$$

*Then we obtain*

$$b^* = (\cdots, i'_{k_j}, i'_{k_j+1}, \cdots) = (\cdots, 0, 0, \bar{d}_1, \cdots, \bar{d}_{|m|-1}, (-)^{t+1} m, (-)^{t+2} m, \cdots),$$

*where $t$ is the position of the right-most entry in the subsequence $\bar{d}_1, \cdots, \bar{d}_{|m|-1}$, that is, $t = k_j + l_j + l_{j+1} + \cdots + l_{|m|-1} - 1$ if $l_j \neq 0$.*

*Proof of Theorem 5.18.* We assume $m > 0$ and then all walls in $b$ are $-$. Set $b = b_1 \otimes t_\mu \otimes b_2 \in B(\infty) \otimes T_\mu \otimes B(-\infty)$ $(\mu := -\lambda - wt(b_1) - wt(b_2))$, and

$$\begin{aligned} b_1 &= (\cdots, i_k, \cdots, i_{-2}, i_{-1}), \\ t_\mu \otimes b_2 &= (i_0, i_1, \cdots, i_k, \cdots). \end{aligned}$$

Let $m_1$ and $m_2$ be the total number of walls in $b_1$ and $t_\mu \otimes b_2$ respectively. Here note that $m_1 + m_2 \leq m$ since the walls of $b$ at position 0 are not included in $b_1$

nor $t_\mu \otimes b_2$. Let $d^1_j$ $(1 \le j \le m_1)$ and $d^2_j$ $(1 \le j \le m_2)$ be the domain in $b_1$ and $t_\mu \otimes b_2$ respectively and set $l^1_j$ and $l^2_j$ the length of $d^1_j$ and $d^2_j$ respectively and $k^1_j$ and $k^2_j$ the position of the left-most entry in $d^1_j$ and $d^2_j$ with non-zero length respectively. That is,

$$\begin{aligned} b_1 &= (\cdots, 0, 0, d^1_1, d^1_2, \cdots, d^1_{m_1}), \\ t_\mu \otimes b_2 &= (d^2_{m_2}, d^2_{m_2-1}, \cdots, d^2_1, 0, 0, \cdots). \end{aligned}$$

Here note that $l^1_j = l_j$ for $(1 \le j < m_1)$ and $l^2_j = l_{m+1-j}$ for $(1 \le j < m_2)$. Also note that $k^1_j = k_j$ if $l^1_j > 0$ $(1 \le j \le m_1)$ and $k^2_j = k_{m+1-j}$ if $l^2_j > 0$ $(1 \le j < m_2)$. By Proposition 5.12, Proposition 5.17 and (5.44), we have

$$b^*_1 = (\cdots, 0, 0, \hat{d}^1_1, \cdots, \hat{d}^1_{m_1}), \tag{5.47}$$

$$t_\lambda \otimes b^*_2 = (\hat{d}^2_{m_2}, \hat{d}^2_{m_2-1}, \cdots, \hat{d}^2_1, (-)^{t+1}m, (-)^{t+2}m, \cdots), \tag{5.48}$$

where $t$ is the position of the right-most entry in the subsequence $\hat{d}^2_{m_2}, \hat{d}^2_{m_2-1}, \cdots, \hat{d}^2_1$ and

$$\hat{d}^1_j = \underbrace{(-)^{k^1_j} j, (-)^{k^1_j+1} j, \cdots, (-)^{k^1_{j+1}-1} j}_{l^1_j}, \tag{5.49}$$

$$\hat{d}^2_j = \underbrace{(-)^{k^2_j}(m-j), (-)^{k^2_j+1}(m-j), \cdots, (-)^{k^2_{j-1}-1}(m-j)}_{l^2_j}. \tag{5.50}$$

In particular, if we denote $b^* = b^*_1 \otimes t_\lambda \otimes b^*_2 = (\cdots, j_k, j_{k+1}, \cdots)$, we have

$$j_{-1} = -m_1 \quad \text{and} \quad j_0 = m - m_2. \tag{5.51}$$

This implies there are $j_{-1} + j_0 = m - m_1 - m_2 \ge 0$ walls at position 0. We know that there are $m_1$ walls in $b^*_1$ and $m_2$ walls in $t_\lambda \otimes b^*_2$. Thus, the total number of walls in $b^*$ is $(m_1 + m_2) + (m - m_1 - m_2) = m$ and there are $m - 1$ finite domains in $b^*$. We denote them $\hat{d}_j$ $(1 \le j \le m-1)$. There are the following two cases:

**(I)** There is no wall at position 0 in $b$ (that is, $m_1 + m_2 = m$).

**(II)** There are walls at position 0 in $b$ (that is, $m_1 + m_2 < m$).

(I) In this case, $m_1 = m - m_2$. Then by (5.51) we know that $j_{-1} + j_0 = 0$ and then there is no wall at position 0 in $b^*$. Thus, we have

$$\begin{aligned} &\hat{d}_j = \hat{d}^1_j = \bar{d}_j \quad (1 \le j < m_1) \quad \text{and} \quad \hat{d}_j = \hat{d}^2_{m-j} = \bar{d}_j \quad (m - m_2 = m_1 < j \le m-1), \\ &\hat{d}_{m_1} = \hat{d}^1_{m_1} \cup \hat{d}^2_{m_2} = (-)^{k_{m_1}} m_1, (-)^{k_{m_1}+1} m_1, \cdots, (-)^{k_{m_1+1}-1} m_1 = \bar{d}_{m_1}. \end{aligned}$$

(II) In this case, $m - m_1 - m_2 > 0$. This menas that there exist $m - m_1 - m_2$ walls at positin 0 in $b^*$ by (5.51).

$$\hat{d}_j = \hat{d}^1_j = \bar{d}_j \quad (1 \le j < m_1) \quad \text{and} \quad \hat{d}_j = \hat{d}^2_{m-j} = \bar{d}_j \quad (m - m_2 < j \le m-1),$$
$$\hat{d}_j = \emptyset = \bar{d}_j \quad (m_1 \le j \le m - m_2).$$

Now, we obtain the desired result. The case $m < 0$ is also shown by the simialr way. $\square$

**Example 5.19** *For* $b = (\cdots, 0, 0, \underbrace{\overset{-5}{-1}, \overset{-4}{1}}_{d_1}, \underbrace{\overset{-3}{-2}, \overset{-2}{2}, \overset{-1}{-2}}_{d_2}, \underbrace{\overset{0}{1}, \overset{1}{-1}, \overset{2}{1}}_{d_3}, \overset{3}{-2}, \overset{4}{2}, -2, 2, \cdots),$ *we have*

$$b^* = (\cdots, 0, 0, \underbrace{\overset{-5}{-1}, \overset{-4}{1}}_{\bar{d}_1}, \underbrace{\overset{-3}{-2}, \overset{-2}{2}, \overset{-1}{-2}}_{\bar{d}_2}, \underbrace{\overset{0}{3}, \overset{1}{-3}, \overset{2}{3}}_{\bar{d}_3}, \overset{3}{-4}, \overset{4}{4}, -4, 4, \cdots).$$

**Corollary 5.20** *For level 0 weight* $\lambda = m(\Lambda_0 - \Lambda_1) + l\delta$ $(l, m \in \mathbf{Z})$,

$$B^{\max}(\lambda) = \bigoplus_{\vec{c} \in \mathbf{Z}^{|m|-1}_{\ge 0}} \mathcal{P}_{m,l}(|m|; \vec{m}; \vec{c}), \tag{5.52}$$

*where*

$$\vec{m} = \begin{cases} (1, 2, \cdots, m-1) & \text{if} \quad m \ge 0, \\ (-1, -2, \cdots, m+1) & \text{if} \quad m < 0. \end{cases}$$

*Proof.* Let $b$ and $\{\bar{d}_j\}$ be as in Theorem 5.18. By (5.45) $t(\bar{d}_j)$ = the type of domain $\bar{d}_j$ (see [10, Sec 7.1]) is given by

$$t(\bar{d}_j) = \begin{cases} j & \text{if} \quad m > 0, \\ -j & \text{if} \quad m < 0. \end{cases}$$

For $b^*$ as in Theorem 5.18 setting $c_j := [[l_j/2]]$ ($[[n]]$ is the Gauss's symbol), we get

$$b^* \in \mathcal{P}_m(|m|; \vec{m}; \vec{c}), \tag{5.53}$$

where $\vec{c} = (c_1, c_2, \cdots, c_{|m|-1})$. This means

$$B^{\max}(\lambda) \subset \bigoplus_{\vec{c} \in \mathbf{Z}^{|m|-1}_{\ge 0}} \mathcal{P}_{m,l}(|m|; \vec{m}; \vec{c}).$$

Now, we assume that $m > 0$ (resp. $m < 0$). Let $b_0$ be an extremal vector in $\mathcal{P}_{m,r}(|m|; \vec{m}; \vec{c})$ $(r \in \mathbf{Z})$ with $-$ walls (resp. $+$ walls) and weight $wt(b_0) = -m(\Lambda_0 - \Lambda_1) - l\delta$. Then $b_0^*$ is an element of $\mathcal{P}_{m,l}$.

This vector $b_0$ is in the following form;

$$b_0 = (\cdots, 0, 0, d_1, d_2, \cdots, d_{|m|-1}, (-1)^{t+1}m, (-1)^{t+2}m, \cdots), \quad d_j = \underbrace{-j, j, -j, \cdots}_{2c_j},$$

where $t$ is the same one as in Theorem 5.18. Because of the form of $d_j$ and the fact $t(d_j) = j$, the position of the left-most entry in any domain is odd. Then by Theorem 5.18, we can write

$$b_0^* = (\cdots, 0, 0, d'_1, d'_2, \cdots, d'_{|m|-1}, (-1)^{t+1}m, (-1)^{t+2}m, \cdots), \quad d'_j = \underbrace{-j, j, -j, \cdots}_{2c_j}.$$

Thus, we have $d_j = d'_j$ and then $b_0 = b_0^*$ as an element of $\mathcal{P}_m(|m|; \vec{m}; \vec{c})$. (Note that in general, $wt(b_0) - wt(b_0^*) \in \mathbf{Z}\delta$.).
Since $\mathcal{P}_{m,l}(|m|; \vec{m}; \vec{c})$ is generated by the extremal vector $b_0^*$ and $b_0^* \in B^{\max}(\lambda)$, we get

$$\mathcal{P}_{m,l}(|m|; \vec{m}; \vec{c}) \subset B^{\max}(\lambda) \quad \text{for any} \quad \vec{c}.$$

Now, we get the desired result. □

## 5.3 Description of $B(-\lambda)^*$

For $\lambda = m(\Lambda_0 - \Lambda_1) + l\delta$ $(m, l \in \mathbf{Z})$, the generator $u_{-\lambda} = u_\infty \otimes t_{-\lambda} \otimes u_{-\infty}$ corresponds to the path

$$b_0 = (\cdots, \overset{-2}{0}, \overset{-1}{0}, \overset{0}{-m}, \overset{1}{m}, \overset{2}{-m}, \cdots).$$

We set

$$\vec{0} := \underbrace{(0, 0, \cdots, 0)}_{|m|-1}.$$

Then $b_0$ is an element of $\mathcal{P}_{-m,-l}(|m|; -\vec{m}; \vec{0})$. This implies

$$B(-\lambda) \cong \mathcal{P}_{-m,-l}(|m|; -\vec{m}; \vec{0}).$$

As we know by the results in the previous subsection, $B(-\lambda) \subset B^{\max}(-\lambda)$. Thus, every element in $B(-\lambda)^*$ is an extremal vector with weight $\lambda$. For an element $b \in \mathcal{P}_{-m,-l}(|m|; -\vec{m}; \vec{0})$, let $d_1(b), d_2(b), \cdots, d_{|m|-1}(b)$ be its finite domains and $l_1(b), l_2(b), \cdots, l_{|m|-1}(b)$ be their lengths. Since any domain in $\mathcal{P}_{-m,-l}(|m|; -\vec{m}; \vec{0})$ is a regular domain (see [10, Sec 7.1]), recalling the definition of domain parameter ([10, Sec 7.1]), we have $l(d_j) = 0$ or $1$ for any $j$, namely,

$$\mathcal{P}_{-m,-l}(|m|; -\vec{m}; \vec{0}) = \{b \in \mathcal{P}_{m,-l}(|m|) \,|\, l(d_j) = 0 \text{ or } 1 \text{ for } j = 1, 2, \cdots, |m| - 1\}. \tag{5.54}$$

We can describe the action of $*$ on an element in $B(\lambda) \subset B^{\max}(\lambda)$ by Theorem 5.18 because $*^{-1} = *$. Then we obtain the following;

**Proposition 5.21** *For $\lambda = m(\Lambda_0 - \Lambda_1) + l\delta$,*

$$B(-\lambda)^* = \left\{ b \in B(\tilde{U}_q(\widehat{\mathfrak{sl}_2})_0) \middle| \begin{array}{c} b \text{ is an extremal vector with } |m| \text{ walls, } wt(b) = \lambda, \\ l_j(b) = 0 \text{ or } 1 \text{ for any } j = 1, \cdots, |m| - 1 \end{array} \right\}. \tag{5.55}$$

## 5.4 Explicit form of Peter-Weyl type decomposition of $B(\tilde{U}_q(\widehat{\mathfrak{sl}_2})_0)$

The crystal of modified quantum algebra $B(\tilde{U}_q(\widehat{\mathfrak{sl}_2}))$ has a decomposition of bi-crystal;

$$B(\tilde{U}_q(\widehat{\mathfrak{sl}_2})) = B(\tilde{U}_q(\widehat{\mathfrak{sl}_2})_+) \oplus B(\tilde{U}_q(\widehat{\mathfrak{sl}_2})_0) \oplus B(\tilde{U}_q(\widehat{\mathfrak{sl}_2})_-),$$

where $\tilde{U}_q(\widehat{\mathfrak{sl}_2})_\pm := \sum_{\pm\langle c,\lambda\rangle>0} U_q(\widehat{\mathfrak{sl}_2})a_\lambda$. The crystallized Peter-Weyl type decomositios of $B(\tilde{U}_q(\widehat{\mathfrak{sl}_2})_\pm)$ have been given in [4]. By applying Proposition 4.4, we can describe the crystallized Peter-Weyl type decomposition for $B(\tilde{U}_q(\widehat{\mathfrak{sl}_2})_0)$.

**Theorem 5.22** *There exists the following isomorphism of bi-crystal;*

$$B(\tilde{U}_q(\widehat{\mathfrak{sl}_2})_0) \cong \bigoplus_{\lambda \in P_0/W} B^{\max}(\lambda) \otimes B(-\lambda)^*, \tag{5.56}$$

*where $P_0 = \{\lambda \in P | \langle c, \lambda\rangle = 0\}$ and $W$ is the Weyl group associated with $\widehat{\mathfrak{sl}_2}$.*

*Proof.* In the course of the proof of Proposition 4.4, it is enough to show that the following conditions hold:

**(C1')** For any extremal vector $b \in B(\tilde{U}_q(\widehat{\mathfrak{sl}_2})_0)$, there exists an embeding of crystal

$$B(wt(b)) \hookrightarrow B(\tilde{U}_q(\widehat{\mathfrak{sl}_2})_0),$$

given by $u_{wt(b)} \mapsto b$.

**(C2')** For any $\lambda \in P_0$, $B(\lambda)_\lambda = \{u_\lambda\}$.

**(C3')** For any extremal vectors $b_1, b_2 \in B(\lambda)$ ($\lambda \in P_0$) there exist $i_1, i_2, \cdots, i_k$ such that

$$b_2 = S_{i_1} S_{i_2} \cdots S_{i_k} b_1.$$

(C1') For an extremal vector $b \in B(\tilde{U}_q(\widehat{\mathfrak{sl}_2})_0)$ let $B'$ be the connected component including $b$ and set $\lambda = m(\Lambda_0 - \Lambda_1) + l\delta := wt(b)$. By Corollary 7.28 in [10], we have

$$\begin{array}{ccccc} B' & \cong & \mathrm{Aff}(B^{\otimes|m|})_{\bar{l}} & \cong & B(\lambda) \\ b & \leftrightarrow & z^l \otimes (\epsilon)^{\otimes|m|} & \leftrightarrow & u_\lambda, \end{array} \tag{5.57}$$

where $\epsilon \in \{\pm\}$ and $\bar{l} \in \{0, 1, \cdots, |m|-1\}$ and $\bar{l} \equiv l \bmod |m|$. This implies that (C1') holds.

(C2') The vector $u_\lambda$ corresponds to the path

$$b = (\cdots, \overset{-2}{0}, \overset{-1}{0}, \overset{0}{m}, \overset{1}{-m}, \overset{2}{m}, \cdots).$$

This $b$ is an element of $\mathcal{P}_{m,l}(|m|; \vec{m}; \vec{0})$. Thus, we have

$$B(\lambda) \cong \mathcal{P}_{m,l}(|m|; \vec{m}; \vec{0}).$$

By Lemma 7.27 in [10] and the comments below that lemma, we get (C2').

(C3') By the formula (7.47) in [10], we obtain the transitivity of extremal vectors in $B(\lambda)$ for $\lambda \in P_0$. $\square$

**Corollary 5.23** *Let us denote $P_1(m, l; \vec{c})$ for $\mathcal{P}_{m,l}(|m|; \vec{m}; \vec{c})$ and $P_2(m, l)$ for the R.H.S of (5.55). We have*

$$B(\widetilde{U}_q(\widehat{\mathfrak{sl}_2})_0) \cong \bigoplus_{\substack{m(\Lambda_0 - \Lambda_1) + l\delta \in P_0/W \\ \vec{c} \in \mathbf{Z}_{\geq 0}^{|m|-1}}} P_1(m, l; \vec{c}) \otimes P_2(m, l).$$

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